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AN UNSTABLE SOLUTION TO THE PROBLEM OF ADVECTION BY THE FINITE DIFFERENCE EULERIAN METHOD

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André Robert
Central Analysis Office
Department of Transport
Meteorological Service of Canada

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The differential equation

$$\frac{\partial F}{\partial t} + U \frac{\partial F}{\partial x} = 0 \tag{1}$$

is normally integrated with the following finite difference algorithm

$$F(x, t+\Delta t) = F(x, t-\Delta t) - U \frac{\Delta t}{\Delta x}\left[F(x+\Delta x, t) - F(x-\Delta x, t)\right] \tag{2}$$

and it is generally considered that the numerical integration will remain stable for any arbitrary advecting function U provided that the following condition is satisfied

$$\left| U(x, t) \right| \leq \frac{\Delta x}{\Delta t} \tag{3}$$

for all x and all t considered in the calculations.

Experiments with equation (2) where U is provided by a simple model of the primitive equations show that this finite difference equation frequently produces unstable solutions for F.

The values of U produced by the model tend to become decoupled with time. The values at even time steps follow a pattern while the values of U at odd time steps follow a completely different pattern. The growth of F in equation (2) seems to be associated with the decoupling occurring in the advecting function U.

It can easily be shown that equation (3) represents a necessary condition for stability and it is usually implied erroneously that this is also a sufficient condition.

We will show that this condition is not sufficient by giving an unstable example that satisfies equations (2) and (3) simultaneously.

$$U(x, t) = E(-1)^{t/\Delta t} \text{ where } E \text{ is a constant} \tag{4}$$

$$F(x, t) = \left[1 + i\,(-1)^{t/\Delta t}\right] e^{wt + ikx} \tag{5}$$

where w is given by

$$\sin h\ (w\Delta t) = E \frac{\Delta t}{\Delta x} \sin\ (k\ x) \tag{6}$$

Substitution in equation (2) gives

$$\left[1 - i(-1)^{t/\Delta t}\right] (e^{w\Delta t})\ e^{wt + i\,k\,x} =$$

$$-E(-1)^{t/\Delta t}\left(\frac{\Delta t}{\Delta x}\right)\left[1 + i(-1)^{t/\Delta t}\right](e^{i\,k\Delta x} - e^{-i\,k\Delta x})\ e^{wt + ikx}$$

which we may rewrite as

$$\left[1 - i(-1)^{t/\Delta t}\right]\sin h(w\Delta t) = -i(-1)^{t/\Delta t}\ \left(E\,\frac{\Delta t}{\Delta x}\right)\left[1 + i(-1)^{t/\Delta t}\right]\sin\ (k\Delta x)$$

or

$$\begin{aligned} 1 - i(-1)^{t/\Delta t} &= -\,i(-1)^{t/\Delta t}\left[1 + i(-1)^{t/\Delta t}\right] \\ &= -\,i\left[i + (-1)^{t/\Delta t}\right] \\ &= -\ \left[-\,1 + i(-1)^{t/\Delta t}\right] \\ &= 1 - i(-1)^{t/\Delta t} \end{aligned}$$

This substitution shows that the functions defined by equations (4), (5) and (6) are a solution of equation (2). Also if E is taken sufficiently small then equation (3) will be satisfied. On the other hand we find that F defined by equation (5) grows for any positive E. We deduce from this example that the condition given by equation (3) is not sufficient for stability.

We conclude that centered differences in both time and space will sometimes generate growing computational modes even when sufficiently short time steps are used.