U. S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER

OFFICE NOTE 158

Diffusion Coefficient Formulation for the Spectral Model

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AUGUST 1977

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1. Introduction

As a result of the experimental run of J. Sela's spectral model model (30 wave) on the case of 17 Feb 77 (x-contour flow near Hawaii), it appears desirable to use a diffusion operation in the equations. Past experience with the LFM has indicated the desirability of using a non-Fickian form of diffusion, i.e. a form in which the diffusion coefficient is a function of wavelength. Such a formulation is readily developed in a spectral model. In this note, we work out an appropriate value for the diffusion; such that it simulates the behavior of the LFM (see Office Note 129), or attains a maximum consistent value.

2. Development of Formula

Consider two forms for diffusion in the spectral model. In one form the diffusion coefficient is a constant; in the alternate form the diffusion coefficient is a constant times n(n+1).

In one step, the responses are:

$$R_{s_1} = 1 - \frac{2\Delta t\ K_1\ n(n+1)}{a^2} \qquad (1)$$

$$R_{s_2} = 1 - \frac{2\Delta t\ K_2[n(n+1)]^2}{a^2} \qquad (2)$$

We take $\Delta t = 1320$ secs (Sela's value for wave 30 rhomboidal version). The radius of the Earth 6.371×10^6m is denoted by a.

So $$R_{s_1} = 1 - 65.0412 \times 10^{-12}\ K_1\ n(n+1) \qquad (3)$$

$$R_{s_2} = 1 - 65.0412 \times 10^{-12}\ K_2\ [n(n+1)]^2 \qquad (4)$$

For $n = 20$ (wave length ~ 1953 km)

$$R_{s_1}(20) = 1 - 2.73173 \times 10^{-8}\ K_1 \qquad (5)$$

$$R_{s_2}(20) = 1 - 1.14733 \times 10^{-5}\ K_2 \qquad (6)$$

Since it requires 65.45 steps to go forward one day, the response for n=20 at one day is

$$R_{s_1}^{65.45}(20) = [1 - 2.73173 \times 10^{-8} K_1]^{65.45} \quad (7)$$

$$R_{s_2}^{65.45}(20) = [1 - 1.14733 \times 10^{-5} K_2]^{65.45} \quad (8)$$

The LFM filter for this wave length ~ 2000 km, yields a response of 0.7 at one day. For equivalence, one must have

$$K_1 = (1 - .7^{1/65.45}) \div 2.73173 \times 10^{-8} \quad (9)$$

$$K_2 = (1 - .7^{1/65.45}) \div 1.14733 \times 10^{-5} \quad (10)$$

$$(1 - .7^{1/65.45}) = 5.4348 \times 10^{-3}$$

so

$$K_1 = 1.989 \times 10^5 \quad m^2 sec^{-1} \quad (11)$$

$$K_2 = 4.737 \times 10^2 \quad m^2 sec^{-1} \quad (12)$$

Using (11) and (12) in (3) and (4) the latter become

$$R_{s_1} = 1 - 1.29367 \times 10^{-5}\, n(n+1) \quad (13)$$

$$R_{s_2} = 1 - 3.081 \times 10^{-8}\, [n(n+1)]^2 \quad (14)$$

Using (13) and (14) we may tabulate the response after one day (65.45 steps) as n varies:

L(km)	n	$R_{s_1}^{65.45}$	$R_{s_2}^{65.45}$
16,342	2	.994	.999$^+$
8,951	4	.983	.999
6,177	6	.965	.996
4,718	8	.940	.990
3,817	10	.911	.976
2,583	15	.815	.890
1,953	20	.70	.70
1,570	25	.575	.424
1,313	30	.453	.171
988	40	.256	.003
792	50	.111	$< 10^{-3}$
662	60	.042	$< 10^{-3}$

We have plotted the response after one day of the alternate spectral model diffusion relations and of the LFM formulation (cf. Figure).

To the extent that the LFM results to 48 hrs are acceptable, it would appear desirable to use the second formulation in the spectral model

$$K = 4.7 \times 10^2 \quad n(n+1) \quad m^2/sec^{-1} \tag{15}$$

(we note that n is dimensionless!)

It is also abundantly clear that Fickian diffusion is seriously deficient in its scale selectivity.

3. An Alternative Nonlinear Form

Since one has considerable freedom in formulating diffusion in a spectral model, another formulation has been considered. Suppose we write

$$K = \hat{D}(n/\hat{n})^2 \ , \quad \hat{n} \geq n \geq n_c \qquad (16)$$

$$K = 0 \ , \quad n < n_c$$

and set $\hat{D}$ to its maximum allowable value. This would permit very strong diffusion in the highest wave numbers, but cut-off diffusion at the wave number n_c.

The maximum value of $\hat{D}$ for consistency of the "leap-frog" approximation of the spectral equation is found to be

$$\hat{D} = \frac{a^2}{2\Delta t \ \hat{n}(\hat{n}+1)} \qquad (17)$$

Using $\hat{n} = 60$, $t = 1320$, $a = 6.371 \times 10^6$m, one gets

$$\hat{D} = 4. \times 10^6 \ m^2 sec^{-1}, \qquad (18)$$

and from (16) with $\hat{n} = 60$

$$K = 1.1 \times 10^3 n^2 \qquad n \geq n_c \qquad (19)$$

$$K = 0 \qquad n < n_c$$

Comparison of (19) with (15) shows the larger magnitude of this alternative form for waves with $n > n_c$.

Use of (19) in the equation

$$\frac{\partial F}{\partial t} = K\nabla^2 F$$

yields in spectral form,

$$\frac{\partial F_n^\ell}{\partial t} = - 1.1 \times 10^3 \ n^2 \ \frac{n(n+1)}{a^2} \ F_n^\ell, \quad n \geq n_c \qquad (20a)$$

$$= 0 \qquad n < n_c \qquad (20b)$$

The spectral response of this form is

$$R_{S_3} = (1 - 7.2268 \times 10^{-8} \ n^3(n+1)) \qquad n \geq n_c$$

$$R_{S_3} = 1 \qquad n < n_c$$

The response after 24 hrs, 36 hrs, and 48 hrs given by R_{S_3} is tabulated below ($n_c = 0$)

L(km)	n	24	36	48
16,342	2	$.999^+$	$.999^+$	$.999^+$
8,951	4	.998	.998	.997
6,177	6	.993	.989	.985
4,718	8	.978	.968	.957
3,817	10	.949	.925	.901
2,583	15	.774	.681	.599
1,953	20	.450	.301	.202
1,570	25	.142	.054	.002
1,313	30	.017	.002	$< 10^{-3}$

This response (24 hrs) is also shown in the figure.

4. Discussion

We have simply scratched one facet of the question, how should one formulate diffusion in a spectral model? It isn't likely that a spectral model will require diffusion for the same reasons as a finite-difference model; thus, variations in the remedies prescribed for numerical weaknesses ought to be different, in either dosage and/or formulation.

The specifics presented in this note suggest that the LFM's value for diffusion may be suitable for the spectral model. There may be advantages in using a cut-off n_c at n = 15.