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An Approach to the Development of More General Quasi-Geostrophic Models

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Introduction

A series of 120 numerical forecasts computed by GRD in 1954 indicated that large errors were often traceable to:

(a) Errors in arbitrarily specified lateral boundary conditions

(b) Errors in lower boundary condition (effect of large scale irregularities of terrain omitted)

(c) Possibly truncation errors, and

(d) Probably defects of the physical Models themselves.

It was also concluded that genuine differences between physically different models (in this case, the barotropic and two-parameter baroclinic models) were obscured by (a) and (c). At that time, it appeared essential to remove (a) and (c) before (b) and (d) could be effectively isolated. Since then, it has been found that the principal manifestations of truncation error are (1) a systematic underestimate of the bulk displacement of pressure systems and (2) the generation of spurious, but random small-scale fluctuations which eventually contaminate the entire calculation. The first of these effects may be reduced by proceeding to higher order finite-difference approximations, and the second has been effectively removed from short range forecasts by smoothing of the type described by Shuman. The effect of incorrect lateral boundary conditions on the solution in the central portions of the grid has been reduced simply by extending the grid over a very large area.

It has also been found that large systematic errors frequently accrue in cases where they cannot be attributed to truncation and boundary errors. These typically result in overintensification of cyclones and, less characteristically, of cyclones. It commonly happens, in fact, that the circulation around the anticyclones steadily increases to a point where it is quite as strong as around cyclones, ultimately producing highs and lows of very similar "shape". In Nature, of course, this is not the case, for there is a marked asymmetry between the shapes of the cyclones and anticyclones. Turning to the physical models in current use, it is suggestive that the so called process of "linearization" removes the asymmetry that is inherent in the coefficients of the general nonlinear equations. Moreover, one can see qualitatively that the introduction of constant standard values of absolute vorticity and static stability (wherever they enter <u>undifferentiated</u> in the potential vorticity equation) not only has the effect of intensifying the divergence field in anticyclones, but also intensifies the effect of the divergence by overestimating the absolute vorticity. These facts indicate that the present quasi-geostrophic models should be totally or partially "delinearized", a change that is difficult to introduce into a computing program as inflexible as the current operational code. The purpose of this note is to outline a proposed scheme for developing more general quasi-geostrophic models. A convenient feature of this scheme is its flexibility, which permits systematic experiments with various degrees of "delinearization" without major changes of computing program.

ω - equation

[illegible] the [illegible] equation of motion is:

$$\frac{\partial \mathbb{V}}{\partial t} + \nabla\left(\frac{\mathbb{V}\cdot\mathbb{V}}{2}\right) + \mathbb{k}\times\eta\mathbb{V} + \omega\frac{\partial \mathbb{V}}{\partial p} + \nabla\phi = 0$$

where $\mathbb{k}$ is a unit vector directed vertically upward. The geostrophic wind is:

$$\mathbb{V} = \mathbb{k}\times\frac{1}{f}\nabla\phi \quad . \quad \text{whence}$$

$$\frac{\partial \mathbb{V}}{\partial p} = \mathbb{k}\times\frac{1}{f}\nabla\frac{\partial\phi}{\partial p} = -\mathbb{k}\times\frac{1}{f}\nabla\alpha$$

where α is the specific volume. Substituting in the [illegible] equation [illegible],

$$\frac{\partial \mathbb{V}}{\partial t} + \nabla\left(\frac{\mathbb{V}\cdot\mathbb{V}}{2}\right) + \mathbb{k}\times\eta\mathbb{V} - \mathbb{k}\times\frac{1}{f}\omega\nabla\alpha + \nabla\phi = 0$$

Applying the curl operator to the equation above,

$$\frac{\partial \eta}{\partial t} + \mathbb{V}\cdot\nabla\eta + \eta\nabla\cdot\mathbb{V} - \frac{1}{f}\nabla\cdot\omega\nabla\alpha = 0 \qquad (1)$$

[illegible] note that $\nabla\cdot\mathbb{V} = -\frac{\partial\omega}{\partial p}$ and that:

$$\theta = \alpha p^{\frac{1}{\gamma}} \qquad \nabla\alpha = p^{-\frac{1}{\gamma}}\nabla\theta$$

[illegible]stituting for $\nabla\alpha$ in Eq. (1) and letting $\Omega = p^{-\frac{1}{\gamma}}\omega$

[illegible] may rewrite Eq (1) as:

$$[illegible] + V\cdot\nabla\eta - \eta\frac{\partial\omega}{\partial p} - \frac{1}{f}\nabla\cdot\Omega\nabla\theta = 0$$

[illegible]y, we differentiate the equation above with respect to p, and multiply by f.

$$[illegible]\frac{\partial}{\partial t}\left(\frac{\partial\zeta}{\partial p}\right) - f\frac{\partial}{\partial p}\left(\eta\frac{\partial\omega}{\partial p}\right) - \frac{\partial}{\partial p}\nabla\cdot\Omega\nabla\theta = -f\frac{\partial}{\partial p}(V\cdot\nabla\eta) \qquad (2)$$

[illegible]

$$\frac{\partial\zeta}{\partial p} = -\frac{1}{f}\nabla^2\alpha$$

$$f\frac{\partial}{\partial t}\left(\frac{\partial\zeta}{\partial p}\right) = -\nabla^2\frac{\partial\alpha}{\partial t}$$

$$= -\nabla^2 p^{-\frac{1}{\gamma}}\frac{\partial\theta}{\partial t}$$

$$= \nabla^2 p^{-\frac{1}{\gamma}}\left(V\cdot\nabla\theta + \frac{\partial\theta}{\partial p}\omega\right)$$

$$= \nabla^2\left(V\cdot\nabla\alpha + \Omega\frac{\partial\theta}{\partial p}\right)$$

or, substituting in Eq. (2),

$$\nabla \cdot \Omega \frac{\partial\theta}{\partial p} - \frac{\partial}{\partial p} \nabla \cdot \Omega \nabla\theta - f \frac{\partial}{\partial p}\left(\eta \frac{\partial\omega}{\partial p}\right) = -\nabla^2 (V \cdot \nabla \chi)$$

$$- f \frac{\partial}{\partial p} V \cdot \nabla \eta = F(x, y, p)$$

The first two terms of the equation above may be written as

$$\nabla \cdot \frac{\partial\theta}{\partial p} \nabla\Omega + \nabla \cdot \Omega \nabla \frac{\partial\theta}{\partial p} - \nabla \cdot \Omega \nabla \frac{\partial\theta}{\partial p} - \nabla \cdot \frac{\partial\Omega}{\partial p} \nabla\theta$$

The second and third terms of this expression cancel, so that

$$\boxed{\nabla \cdot p^{\frac{1}{\gamma}} \frac{\partial\theta}{\partial p} \nabla\omega - \nabla \cdot \frac{\partial\Omega}{\partial p} \nabla\theta - f \frac{\partial\eta}{\partial p} \frac{\partial\omega}{\partial p} - f\eta \frac{\partial^2\omega}{\partial p^2} = F(x, y, p)}$$

This is the general equation for ω (or for $\Omega = p^{-\frac{1}{\gamma}}\omega$)

We now adopt the point of view that the ... stic method yield very accurate values of ω ($\sim \Omega$) ... only where ω ($\sim \Omega$) reaches its maximum value, [illegible] ω [illegible] reaches [illegible] maximum value at [illegible]

[illegible] the [illegible] of the [illegible]

[illegible] at an approximate

that [illegible] accurate values of ω

[illegible] 500 mb, we shall omit only those

[illegible] which [illegible] are necessarily [illegible].)

[illegible] finite-difference equation for ω with

[illegible] 0, 500, and 1000 mb [illegible]

[illegible] that terms involving $\frac{\partial \omega}{\partial p}$ or

$\frac{\partial}{\partial p}$ [illegible], are necessarily small in comparison

with terms in which derivatives of ω with respect

[illegible] of [illegible] second order. Thus, an

[illegible] differential equation that will yield accurate

[illegible] where ω reaches its maximum value [illegible]

$$\nabla \cdot \sigma \nabla \omega - f_n \frac{\partial^2 \omega}{\partial p^2} = F(x, y, p) \qquad (3)$$

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* [illegible], Jan 1953, [illegible]

$$\sigma = p^{-1}\frac{\partial\theta}{\partial p} = \frac{\alpha}{\theta}\frac{\partial\theta}{\partial p}$$

$$= -\frac{RT}{p\theta}\cdot\frac{1}{g\rho}\cdot\frac{\partial\theta}{\partial z}$$

$$= -\frac{R^2T^2}{p^2}\cdot\frac{1}{g\theta}\frac{\partial\theta}{\partial z}$$

Thus, Eq. (3) may be written as

$$\boxed{\nabla\cdot c^2\nabla\omega + f\eta\frac{\partial^2\omega}{\partial p^2} = -F(x,y,p)} \qquad (4)$$

where $c^2 = \frac{R^2T^2}{g\theta}\frac{\partial\theta}{\partial z}$

This equation is of the same type as the equation recently derived by Thompson, and whose numerical solution is now being studied by Cressman and Jones. In the latter case, we have taken constant standard values of [illegible] and c. The reasoning outlined above indicates that the ω-equation now under study at the JNWP[illegible] It contains the most important physical ingredients, and will probably yield the right pattern of ω.

... of the intensities of the maxima and minima are not exactly correct. This lends further justification for feeding back such approximate values of ω into the vorticity equation, in order to investigate the effects of "delinearizing" the terms corresponding to divergence, vertical advection and reorientation of the vortex tubes.

It is proposed, moreover, that the development of more general quasi-geostrophic models be approached through Eq. (4), using the *observed* values of f, η, and c. Even with variable $f\eta$ and c, the ω-equation is linear & elliptic (provided $c > 0$ and $\eta > 0$), and can be solved by standard relaxation methods.

Once the ω-field has been computed, the height changes at each reference level can be computed from the *general* nonlinear vorticity equation (which then takes the form of a Poisson equation) by *two*-dimensional relaxation. i.e. —

$$\frac{1}{c}\nabla^2\left(\frac{\partial z}{\partial t}\right) + V\cdot\nabla\eta - \eta\frac{\partial \omega}{\partial p} - \frac{1}{f}\nabla\cdot\omega\nabla\alpha = 0 \qquad (5)$$

procedure is essentially that outlined earlier by the [illegible] to solve [illegible] the equations for the [illegible]-integrated" model.

Admittedly, Eq. (4) is not exact. However, since the atmosphere behaves as a barotropic fluid to a fair degree of approximation, it is evident that the ω-field need not be known as accurately as the height-tendency field (the equation for barotropic flow is simply Eq. (5) with all terms involving ω omitted). <u>Loosely speaking</u>, all that is required to improve on the barotropic model is to compute an ω-field that is better than a uniform guess of $\omega = 0$. From this point of view, we are probably allowed certain liberties in computing ω that cannot be taken in computing height changes. It is conceivable, for example, that the ω-equation can be "linearized" — to the extent of assigning standard values of η and c — but the height-tendency (vorticity) equation cannot. Experiments now under way should settle this question.

There is another point involved in this particular approach to the "general" quasi-geostrophic model that transcends questions as to the order in which the calculations are to be carried out. That is simply that the physical boundary conditions are essentially kinematical constraints on ω, rather than dynamical constraints on the pressure (or contour height). The latter have been found to be extremely sensitive to errors in the prediction of temperature change at the physical boundaries, whereas the ω-field is not particularly sensitive to errors in specifying ω at the lower boundary. In my opinion, based partly on an analysis of the effect of physical boundary conditions and partly on nothing but sheer intuition, the approach through a diagnostic ω-equation makes the best sense from a computational standpoint. Here, at least, the upper and lower boundary conditions at each time stage are not so crucially dependent on the forecast from a previous time stage.