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ON AN IMPROVED FORM OF THE HYDROSTATIC EQUATION IN
FINITE DIFFERENCES

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The horizontal and vertical equations of motion may be written in σ -coordinates as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \dot{\sigma} \frac{\partial u}{\partial \sigma} + \frac{\partial \phi}{\partial x} + c_p \theta \frac{\partial \pi}{\partial x} = 0, \quad (1)$$

$$\frac{\partial \phi}{\partial \sigma} + c_p \theta \frac{\partial \pi}{\partial \sigma} = 0. \quad (2)$$

The vorticity equation formed from these is

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \dot{\sigma} \frac{\partial}{\partial \sigma} \right) \frac{\partial u}{\partial \sigma} + \frac{\partial u}{\partial \sigma} \left(\frac{\partial u}{\partial x} + \frac{\partial \dot{\sigma}}{\partial \sigma} \right) + c_p \left(\frac{\partial \theta}{\partial \sigma} \frac{\partial \pi}{\partial x} - \frac{\partial \theta}{\partial x} \frac{\partial \pi}{\partial \sigma} \right) = 0 \quad (3)$$

Attention is directed to the last term, the so-called solenoidal term which represents a conversion due to baroclinicity. In finite differences, we would like to have the form of this important term preserved. It will first be shown that the present finite difference counterparts to (1) and (2) used in the NMC six-layer primitive equation model do not preserve the form of the solenoidal term when the vorticity equation is formed, and secondly a simple remedy will be shown.

Neglecting variations in the y-direction, map factor, and overriding operators, the horizontal and vertical pressure force terms of the six-layer PE model are written

$$P_x = \overline{\phi_x}^\sigma + c_p \overline{\theta^{\overline{x-\sigma}} \pi_x}^\sigma \quad (4)$$

and

$$P_\sigma = \phi_\sigma + c_p \theta \pi_\sigma, \quad (5)$$

where standard notation is used for the operators. The solenoidal term in the vorticity equation is formed from these by taking

$$\begin{aligned} & (P_x)_\sigma - \overline{(P_\sigma)_x}^\sigma \\ &= \left(\overline{\phi_x}^\sigma + c_p \overline{\theta^{\overline{x-\sigma}} \pi_x}^\sigma \right)_\sigma - \overline{\left(\phi_\sigma + c_p \theta \pi_\sigma \right)_x}^\sigma \\ &= \left(\overline{\phi_{x\sigma}}^\sigma + c_p \overline{\theta^{\overline{x-\sigma}} \pi_{x\sigma}}^\sigma + c_p \overline{\theta^{\overline{x-\sigma}} \pi_{\sigma x}}^\sigma \right) - \left(\overline{\phi_{x\sigma}}^\sigma + c_p \overline{\theta^{\overline{x-\sigma}} \pi_\sigma}^\sigma + c_p \overline{\theta^{\overline{x-\sigma}} \pi_{\sigma x}}^\sigma \right) \\ &= \left(c_p \overline{\theta^{\overline{x-\sigma}} \pi_{\sigma x}}^\sigma - c_p \overline{\theta^{\overline{x-\sigma}} \pi_{x\sigma}}^\sigma \right) + c_p \left(\overline{\theta^{\overline{x-\sigma}} \pi_\sigma}^\sigma - \overline{\theta^{\overline{x-\sigma}} \pi_{\sigma x}}^\sigma \right) \end{aligned} \quad (6)$$

In order for the result to look more like the solenoidal term in eqn. (3), we would need the first term to cancel and the second to have a more consistent form. This may be accomplished by substituting for eqn. (5) the following finite difference form of the hydrostatic equation:

$$P_{\sigma} = \bar{\phi}_{\sigma} + c_p \bar{\theta}^{\sigma} \bar{\pi}_{\sigma} = 0 \quad (7)$$

With this form, the solenoidal term is formed as follows:

$$\begin{aligned} & (P_x)_{\sigma} - (P_{\sigma})_x \\ &= \left(\bar{\phi}_x^{\sigma} + c_p \bar{\theta}^{\sigma} \bar{\pi}_x^{\sigma} \right)_{\sigma} - \left(\bar{\phi}_{\sigma}^{\sigma} + c_p \bar{\theta}^{\sigma} \bar{\pi}_{\sigma}^{\sigma} \right)_x \\ &= \left(\bar{\phi}_{x\sigma}^{\sigma} + c_p \bar{\theta}_{\sigma}^{\sigma} \bar{\pi}_{x\sigma}^{\sigma} + c_p \bar{\theta}^{\sigma} \bar{\pi}_{\sigma x}^{\sigma} \right) - \left(\bar{\phi}_{x\sigma}^{\sigma} + c_p \bar{\theta}_{x\sigma}^{\sigma} \bar{\pi}_{\sigma}^{\sigma} + c_p \bar{\theta}^{\sigma} \bar{\pi}_{\sigma x}^{\sigma} \right) \\ &= c_p \left(\bar{\theta}_{\sigma}^{\sigma} \bar{\pi}_{x\sigma}^{\sigma} - \bar{\theta}_{x\sigma}^{\sigma} \bar{\pi}_{\sigma}^{\sigma} \right) \end{aligned} \quad (8)$$

which has the required form.

Unfortunately, we may not simply replace eqn. (7) for eqn. (5). This is because eqn. (7) expresses the differences of ϕ two-grid increments apart. It seems inevitable to use eqn. (5) to define the difference between first two values of ϕ as shown schematically in Figure 1. A test of the revised eqn. (7) shows a slight improvement.

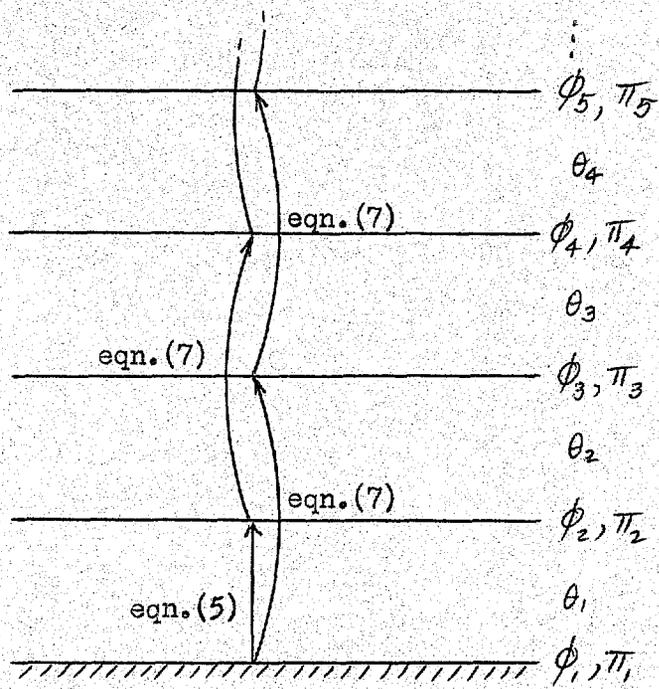


Figure 1.