

U. S. DEPARTMENT OF COMMERCE
ENVIRONMENTAL SCIENCE SERVICES ADMINISTRATION
WEATHER BUREAU

OFFICE NOTE 39

ENERGY EQUATIONS

FOR THE NMC OPERATIONAL PRIMITIVE EQUATION MODEL

by

John A. Brown, Jr.
National Meteorological Center
Suitland, Maryland

February 1970

This note is a description of the energy equations which are appropriate to the σ -system. Also included here is a discussion of energy conservation as it is related to the vertical differencing method employed in the operational model.

In sigma coordinates, we have the following set of differential equations:

$$u_t = -\left(\frac{u^2 + v^2}{2}\right)_x + v\eta - \dot{\sigma} u_\sigma - \varphi_x - c_p \theta \pi_x + F(x) \quad (1)$$

$$v_t = -\left(\frac{u^2 + v^2}{2}\right)_y - u\eta - \dot{\sigma} v_\sigma - \varphi_y - c_p \theta \pi_y + F(y) \quad (2)$$

$$\varphi_\sigma + c_p \theta \pi_\sigma = 0 \quad (3)$$

$$\theta_t = -u\theta_x - v\theta_y - \dot{\sigma} \theta_\sigma + \frac{H}{c_p \pi} \quad (4)$$

$$(p_\sigma)_t = -(u p_\sigma)_x - (v p_\sigma)_y - (\dot{\sigma} p_\sigma)_\sigma \quad (5)$$

$$\pi \theta = T \quad (6)$$

$$\pi = (p/p_0)^\kappa ; \quad \kappa = \frac{R}{c_p} = 2/7 \quad (7)$$

Here

$$\eta = (v_x - u_y) + f$$

H = diabatic heating rate per unit mass,

and $F(x,y)$ = frictional components.

Kinetic Energy Equation

Multiply (1) by $(u p_\sigma)$, (2) by $(v p_\sigma)$, add result and rearrange. One can obtain

$$\left[\left(\frac{u^2 + v^2}{2}\right) p_\sigma\right]_t = \varphi_\sigma \omega - (\varphi p_t)_\sigma + [uF(x) + vF(y)] p_\sigma - \left(m^2 \frac{\partial}{\partial x} \frac{u}{m} + m^2 \frac{\partial}{\partial y} \frac{v}{m} + \frac{\partial}{\partial \sigma} \dot{\sigma}\right) \left[\left(\frac{u^2 + v^2}{2} + \varphi\right) p_\sigma\right],$$

where $\omega = \frac{dp}{dt}$, Here we have now included the map factor, m , Performing the operation

$$\frac{1}{A} \int_A \int_\sigma () \frac{d\sigma}{g} dA$$

yields (A = area)

$$\frac{\partial K}{\partial t} = C(P, K) + D(K) + B(K), \quad (8)$$

where

$$K = \frac{1}{A} \int_A \int_\sigma \frac{u^2 + v^2}{2} p_\sigma \frac{d\sigma}{g} dA \quad (8a)$$

$$C(P, K) = \frac{1}{A} \int_A \int_\sigma [c_p \omega - (\varphi p_t)_\sigma] \frac{d\sigma}{g} dA \quad (8b)$$

$$D(K) = \frac{1}{A} \int_A \int_\sigma [uF(x) + vF(y)] p_\sigma \frac{d\sigma}{g} dA \quad (8c)$$

$$B(K) = -\frac{1}{A} \int_A \int_\sigma \left(m^2 \frac{\partial}{\partial x} \frac{u}{m} + m^2 \frac{\partial}{\partial y} \frac{v}{m} + \frac{\partial}{\partial \sigma} \dot{\sigma}\right) \left[\left(\frac{u^2 + v^2}{2} + \varphi\right) p_\sigma\right] \frac{d\sigma}{g} dA. \quad (8d)$$

K is the kinetic energy per unit area. $C(P,K)$ represents the conversion of potential plus internal energy (hereafter referred to as total potential energy) into kinetic energy. $D(K)$ is the kinetic energy dissipation. $B(K)$ are the kinetic energy equation vertical and lateral boundary terms. It is desirable, of course, to formulate boundary conditions such that this term vanishes.

Total Potential Energy Equation

Multiply equation (4) by $(c_v \pi p_\sigma)$. One obtains

$$(c_v T p_\sigma)_t = - (u c_v T p_\sigma)_x - (v c_v T p_\sigma)_y - (\dot{\sigma} c_v T p_\sigma)_\sigma + \frac{c_v}{p} H p_\sigma + c_v \Theta \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial \sigma} \dot{\sigma} \right) \pi p_\sigma. \quad (9)$$

But $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial \sigma} \dot{\sigma} \right) \pi p_\sigma = p_\sigma \frac{d\pi}{dt}$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \dot{\sigma} \frac{\partial}{\partial \sigma}$.

Since $\frac{d\pi}{dt} = \kappa \frac{\pi}{p} \omega$, the internal energy equation is

$$(c_v T p_\sigma)_t = - \frac{c_v}{p} \varphi_\sigma \omega + \frac{c_v}{p} H p_\sigma - \left(\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial \sigma} \dot{\sigma} \right) (c_v T p_\sigma)_\sigma. \quad (10)$$

Now multiply equation (10) by $\frac{c_p}{c_v}$ and add the following to both sides of the equation:

$$\begin{aligned} (\varphi p_\sigma)_t - (R T p_\sigma)_t &= (\varphi p_\sigma + p \varphi_\sigma)_t \\ &= (\varphi p_t)_\sigma + (p \varphi_t)_\sigma. \end{aligned}$$

We obtain

$$\begin{aligned} [(c_v T + \varphi) p_\sigma]_t &= - \varphi_\sigma \omega + (\varphi p_t)_\sigma + H p_\sigma \\ &\quad - \left(m^2 \frac{\partial}{\partial x} \frac{u}{m} + m^2 \frac{\partial}{\partial y} \frac{v}{m} + \frac{\partial}{\partial \sigma} \dot{\sigma} \right) (c_p T p_\sigma)_\sigma + (p \varphi_t)_\sigma. \end{aligned} \quad (11)$$

Finally upon integrating as before,

$$\frac{1}{A} \int_A \int_\sigma () \frac{d\sigma}{g} dA,$$

we obtain the equation for potential plus internal energy

$$\frac{\partial P}{\partial t} = - C(P,K) + G(P) + B(P). \quad (12)$$

Here

$$P = - \frac{1}{A} \int_A \int_\sigma [(c_v T + \varphi) p_\sigma] \frac{d\sigma}{g} dA \quad (12a)$$

$$G(P) = \frac{1}{A} \int_A \int_\sigma H p_\sigma \frac{d\sigma}{g} dA \quad (12b)$$

$$B(P) = - \frac{1}{A} \int_A \int_\sigma \left[\left(m^2 \frac{\partial}{\partial x} \frac{u}{m} + m^2 \frac{\partial}{\partial y} \frac{v}{m} + \frac{\partial}{\partial \sigma} \dot{\sigma} \right) c_p T p_\sigma - (p \varphi_t)_\sigma \right] \frac{d\sigma}{g} dA. \quad (12c)$$

P is the total potential energy per unit area.

G(P) represents the rate of generation of total potential energy due to diabatic effects.

B(P) represents the lateral and vertical boundary terms. Again it is desirable to formulate the boundary conditions so that these terms vanish.

Vertical Differencing in the PE Model and Energy Conservation

We will now consider the vertical differencing as used in the operational baroclinic PE model in light of the kinetic energy equation (8) and the total potential energy equation (12). It is desirable to use differencing schemes which do not introduce spurious energy sources or sinks. Thus the finite difference forms of terms like

$$-\frac{1}{A} \int_A \int_{\sigma} \left[\dot{\sigma} \left(\frac{u^2 + v^2}{2} + \varphi \right) p_{\sigma} \right]_{\sigma} \frac{d\sigma}{g} dA,$$

which exist in equation (8), and terms like

$$-\frac{1}{A} \int_A \int_{\sigma} \left[\dot{\sigma} c_p T p_{\sigma} - p \varphi_t \right]_{\sigma} \frac{d\sigma}{g} dA,$$

which occur in equation (12), should vanish. The differential forms vanish since $\dot{\sigma} = 0$ at top and bottom, and $p = 0$ at the top, and $\varphi_t = 0$ at the bottom. Using Shuman's notation for the vertical differencing, we have

$$u_t = - \left(\frac{u^2 + v^2}{2} \right)_x + v \eta - \overline{\dot{\sigma} u_{\sigma}} - \overline{\varphi_x^{\sigma}} - c_p \theta \overline{\pi_x^{\sigma}} + F(x) \quad (20)$$

$$v_t = - \left(\frac{u^2 + v^2}{2} \right)_y - u \eta - \overline{\dot{\sigma} v_{\sigma}} - \overline{\varphi_y^{\sigma}} - c_p \theta \overline{\pi_y^{\sigma}} + F(y) \quad (21)$$

$$\varphi_{\sigma} + c_p \theta \overline{\pi_{\sigma}} = 0 \quad (22)$$

$$\theta_t = - u \theta_x - v \theta_y - \overline{\dot{\sigma} \theta_{\sigma}} + H / (c_p \overline{\pi^{\sigma}}) \quad (23)$$

$$(p_{\sigma})_t = - (u p_{\sigma})_x - (v p_{\sigma})_y - \overline{\dot{\sigma} p_{\sigma}}. \quad (24)$$

A. Let us first look at the kinetic energy equation. Multiply (20) by $(u p_{\sigma})$, (21) by $(v p_{\sigma})$ and add.

$$\begin{aligned} \left[\left(\frac{u^2 + v^2}{2} \right) p_{\sigma} \right]_t &= - \left(\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v \right) \left[\left(\frac{u^2 + v^2}{2} \right) p_{\sigma} \right] - u p_{\sigma} \overline{\dot{\sigma} u_{\sigma}} - v p_{\sigma} \overline{\dot{\sigma} v_{\sigma}} \\ &\quad - u p_{\sigma} \overline{\varphi_x^{\sigma}} - v p_{\sigma} \overline{\varphi_y^{\sigma}} - u p_{\sigma} c_p \theta \overline{\pi_x^{\sigma}} - v p_{\sigma} c_p \theta \overline{\pi_y^{\sigma}} \\ &\quad + p_{\sigma} [u F(x) + v F(y)] + \underbrace{\frac{u^2 + v^2}{2} [(p_{\sigma})_t + (u p_{\sigma})_x + (v p_{\sigma})_y]}_{- \overline{\dot{\sigma} p_{\sigma}}} \end{aligned} \quad (25)$$

Consider first the pressure gradient terms.

$$\begin{aligned}
 PG &= -u p_\sigma \bar{\varphi}_x^\sigma - v p_\sigma \bar{\varphi}_y^\sigma - u p_\sigma c_p \theta \bar{\pi}_x^\sigma - v p_\sigma c_p \theta \bar{\pi}_y^\sigma \\
 &= -(u p_\sigma \bar{\varphi}^\sigma)_x - (v p_\sigma \bar{\varphi}^\sigma)_y - c_p \theta p_\sigma (u \bar{\pi}_x^\sigma + v \bar{\pi}_y^\sigma) \\
 &\quad + \bar{\varphi}^\sigma [(u p_\sigma)_x + (v p_\sigma)_y]
 \end{aligned}$$

Now
$$\begin{aligned}
 \bar{\varphi}^\sigma [(u p_\sigma)_x + (v p_\sigma)_y] &= -\bar{\varphi}^\sigma [(p_t)_\sigma + (\dot{\sigma} p_\sigma)_\sigma] \\
 &= -(\varphi p_t)_\sigma + \bar{p}_t^\sigma \varphi_\sigma - p_\sigma (\varphi \dot{\sigma})_\sigma + p_\sigma \dot{\sigma}^\sigma \varphi_\sigma.
 \end{aligned}$$

Let us introduce the following differencing method for a layer:

$$\begin{aligned}
 \omega &\equiv p_t + u p_x + v p_y + \dot{\sigma} p_\sigma \\
 &= \bar{p}_t^\sigma - \frac{c_p \theta}{\varphi_p} (u \bar{\pi}_x^\sigma + v \bar{\pi}_y^\sigma) + \dot{\sigma}^\sigma p_\sigma
 \end{aligned} \tag{26}$$

or

$$-c_p \theta p_\sigma (u \bar{\pi}_x^\sigma + v \bar{\pi}_y^\sigma) = \varphi_\sigma (\omega - \bar{p}_t^\sigma - \dot{\sigma}^\sigma p_\sigma).$$

Thus the pressure gradient terms of equation (25) may be written

$$PG = -(u p_\sigma \bar{\varphi}^\sigma)_x - (v p_\sigma \bar{\varphi}^\sigma)_y + [\varphi_\sigma \omega - (\varphi p_t)_\sigma] - (\varphi \dot{\sigma} p_\sigma)_\sigma.$$

Now equation (25) may be written

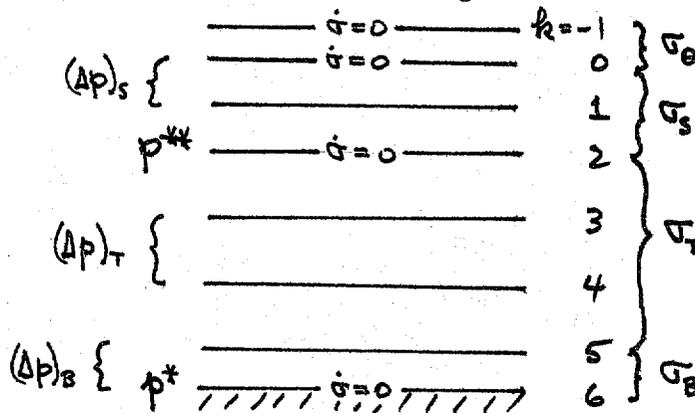
$$\begin{aligned}
 \left[\left(\frac{u^2 + v^2}{2} \right) p_\sigma \right]_t &= [\varphi_\sigma \omega - (\varphi p_t)_\sigma] + [u F(x) + v F(y)] p_\sigma \\
 &\quad - \left(\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v \right) \left[\left(\frac{u^2 + v^2}{2} + \varphi \right) p_\sigma \right] \\
 &\quad - u p_\sigma \dot{\sigma}^\sigma u_\sigma - v p_\sigma \dot{\sigma}^\sigma v_\sigma - \frac{u^2 + v^2}{2} p_\sigma \dot{\sigma}^\sigma - (\varphi \dot{\sigma} p_\sigma)_\sigma.
 \end{aligned} \tag{27}$$

Our interest lies in the last four terms which represent the vertical boundary term

$$- \left[\dot{\sigma} \left[\frac{u^2 + v^2}{2} + \varphi \right] p_\sigma \right]_\sigma.$$

Let us see if these vanish when we integrate over σ .

Consider



Let us evaluate the following:

$$VB = - \int_{\sigma} [u p_{\sigma} \dot{\sigma} u_{\sigma} + N p_{\sigma} \dot{\sigma} N^{\sigma} + \frac{u^2 + N^2}{2} p_{\sigma} \dot{\sigma} + p_{\sigma} (\varphi \dot{\sigma})_{\sigma}] \frac{d\sigma}{g}. \quad (28)$$

Note that in order that $\omega_{SB} = \omega_{ST}$,

$$\dot{\sigma}_{SB} = \frac{3(\Delta p)_T}{(\Delta p)_B} \dot{\sigma}_{ST}. \quad (29)$$

In the operational model it is assumed that

$$(\dot{\sigma} u_{\sigma})_{ST} = \frac{u(\Delta p)_T}{(\Delta p)_B + (\Delta p)_T} \dot{\sigma}_{ST} (u_{5.5} - u_{4.5}) = (\dot{\sigma} u_{\sigma})_{SB} = \frac{2(\Delta p)_B}{(\Delta p)_B + (\Delta p)_T} \dot{\sigma}_{SB} (u_{5.5} - u_{4.5}). \quad (29a)$$

Finally,

$$VB = - \frac{1}{g} \left\{ - \frac{u_{4.5}^2 + u_{5.5}^2}{2} (\Delta p)_T \dot{\sigma}_{ST} \left[\frac{6(\Delta p)_T}{(\Delta p)_B + (\Delta p)_T} - \frac{1}{(\Delta \sigma)_T} \right] + \frac{u_{5.5}^2 + N_{5.5}^2}{2} (\Delta p)_T \dot{\sigma}_{ST} \left[\frac{6(\Delta p)_B}{(\Delta p)_B + (\Delta p)_T} - \frac{3}{(\Delta \sigma)_B} \right] \right. \\ \left. + \frac{3(\Delta p)_T}{(\Delta p)_B + (\Delta p)_T} \dot{\sigma}_{ST} (u_{5.5} u_{4.5} + N_{5.5} N_{4.5}) [(\Delta p)_T - (\Delta p)_B] \right\}$$

But $(\Delta \sigma)_B = 1$ and $(\Delta \sigma)_T = 1/3$. Therefore

$$VB = \left[(u_{5.5} - u_{4.5})^2 + (N_{5.5} - N_{4.5})^2 \right] \frac{3(\Delta p)_B (\Delta p)_T - (\Delta p)_B^2}{2g (\Delta p)_T + (\Delta p)_B} \dot{\sigma}_{ST}. \quad (30)$$

This clearly does not vanish in general. Note that it would vanish if the model had been designed so that $(\Delta p)_B = (\Delta p)_T$.

A spurious kinetic energy source exists when $\dot{\sigma}_{ST} > 0$ and
a spurious kinetic energy sink exists when $\dot{\sigma}_{ST} < 0$.

Had we evaluated (u_{σ}) and (v_{σ}) at the top of the boundary layer in order that $VB = 0$, we would have obtained

$$(u_{\sigma})_{ST} = \frac{3(\Delta p)_T (u_{5.5}^2 - u_{4.5}^2)}{(\Delta p)_T u_{4.5} + (\Delta p)_B u_{5.5}} \quad \text{and} \quad (u_{\sigma})_{SB} = \frac{(\Delta p)_B (u_{5.5}^2 - u_{4.5}^2)}{(\Delta p)_T u_{4.5} + (\Delta p)_B u_{5.5}}$$

rather than

$$(u_{\sigma})_{ST} = \frac{6(\Delta p)_T (u_{5.5} - u_{4.5})}{(\Delta p)_B + (\Delta p)_T} \quad \text{and} \quad (u_{\sigma})_{SB} = \frac{2(\Delta p)_B (u_{5.5} - u_{4.5})}{(\Delta p)_B + (\Delta p)_T}$$

A numerical test was made in which the vertical differencing scheme which allows for $VB = 0$ was incorporated. The model, however, behaved less stably - i.e., a tendency for stratospheric exhaustion to occur early. A possible explanation for this failure is that although this new differencing scheme conserves kinetic energy, it does not conserve momentum. Perhaps a better test of the above analysis would be to evaluate an experiment in which $(\Delta p)_B = (\Delta p)_T$. Thus far, this has not been done.

B. Let us now focus our attention on the vertical differencing used in the thermodynamic energy equation (23). Multiply this equation by $c_p \pi^{\sigma} p_{\sigma}$ and make use of (24) and (26). We obtain the internal energy

equation

$$\begin{aligned}
 (c_v \bar{\pi}^\sigma \theta_{p\sigma})_t = & -\frac{c_v}{c_p} \rho_0 \omega + \frac{c_v}{c_p} \rho_\sigma H - \left[\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v \right] (c_v \bar{\pi}^\sigma \theta_{p\sigma}) \\
 & - c_v \bar{\pi}^\sigma \rho_\sigma \dot{\sigma}_\sigma - c_v \bar{\pi}^\sigma \rho_\sigma \dot{\sigma}_\sigma \theta - c_v \theta_{p\sigma} \bar{\pi}_\sigma \dot{\sigma}^\sigma.
 \end{aligned} \tag{31}$$

Comparing this with (10), we see that the last three terms in (31) must represent $(\dot{\sigma} c_v T p_\sigma)_\sigma$. If we assume that π varies linearly with σ , utilize (29a) and apply to the operational model, we find that the vertical differencing does not conserve the internal energy (or temperature). This can only be done if

$$(\theta_\sigma)_1 = \frac{4\pi_1}{\pi_0 + 2\pi_1 + \pi_2} \frac{\theta_{1.5} - \theta_{0.5}}{(\Delta\sigma)_S}$$

$$(\theta_\sigma)_3 = \frac{4\pi_3}{\pi_2 + 2\pi_3 + \pi_4} \frac{\theta_{2.5} - \theta_{3.5}}{(\Delta\sigma)_T}$$

$$(\theta_\sigma)_4 = \frac{4\pi_4}{\pi_3 + 2\pi_4 + \pi_5} \frac{\theta_{3.5} - \theta_{4.5}}{(\Delta\sigma)_T}$$

$$(\theta_\sigma)_5 = \frac{4\pi_5 (\Delta p)_T}{(\pi_4 + \pi_5)(\Delta p)_T + (\pi_5 + \pi_6)(\Delta p)_B} \frac{\theta_{4.5} - \theta_{5.5}}{(\Delta\sigma)_T}$$

rather than the method in (29b) for θ . However, the present operational vertical finite difference method, as we have discussed it here, conserves θ .