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A DISCRETIZATION OF DEEP-ATMOSPHERIC NONHYDROSTATIC DYNAMICS ON GENERALIZED HYBRID VERTICAL COORDINATES FOR NCEP GLOBAL SPECTRAL MODEL

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ABSTRACT

The deep-atmospheric nonhydrostatic global dynamics are introduced with detailed discretization on spherical and generalized vertical coordinates. Based on the NCEP global spectral model, the horizontal discretization (which is not described in this manuscript) uses the spectral method with spherical spectral transformation; the vertical discretization described in this paper is illustrated in detail up to the level of readiness for programming.

The primitive equations contain three-dimensional momentum, enthalpy as a thermodynamic variable, density, and tracers in height coordinates which are used to convert to generalized vertical coordinates with virtual horizontal winds for spherical coordinates. The equations are examined to show their characteristics of multiple conservations, which are mass conservation, angular momentum conservation, entropy conservation, and total energy conservation.

The concept of mean pressure at any given level by projecting unit air weight on mean earth radius surface is utilized to have a mass coordinate, which results in a similar formulation of the density equation in a hydrostatic system. The mean pressure at a given model level, obtained from the weight concept, is called a coordinate pressure, which has the property of a monotonic decrease with height suitable for the coordinate system.

The angular momentum conservation leads to a discretization for the relationship among coordinate pressure, height, and temperature, which is similar to the hydrostatic relationship in a hydrostatic system, also deduces a relationship for heights between model levels and model layers. The total energy conservation is obtained from three dimensional momentum equations, geopotential height, and the thermodynamic equation. To do total energy conservation, we have a discretization for the total derivative of pressure, which is discretized from the momentum equation and used for the thermodynamic equation, to ensure total energy conservation. The potential enthalpy conservation is also applied to the vertical advection for enthalpy in Eulerian system, which requires multiplying enthalpy to vertical advection of logarithmic enthalpy.

Since sigma-pressure vertical coordinates are used in the NCEP GFS, we give a specific discretization in sigma-pressure hybrid vertical coordinates. The two-time-level semi-implicit semi-Lagrangian scheme is used as example for time integration discretization. The linearization of all prognostic equations is required for the semi-implicit time scheme. The matrices used in the semi-implicit time scheme for linear terms are listed in appendices along with cold start initial fields from the hydrostatic system and detailed derivations for the continuity equation from the height coordinate to generalized hybrid vertical coordinates.

1. Introduction

In EMC (Environmental Modeling Center) of NCEP (National Centers for Environmental Prediction), it is our job to develop numerical models for associated centers within NCEP to use operationally. Not only it is presently a trend to have high-resolution nonhydrostatic global modeling but it is also required as part of the EMC support of the space weather prediction center (SWPC) of NCEP. Thus, a nonhydrostatic and deep atmospheric global model should be considered, so that a global model can be used to support weather and climate for the lower and upper atmosphere and be coupled with other earth system models, such as ocean, ice, and space environment models.

In the literature, a deep-atmospheric nonhydrostatic system on generalized coordinates was given in Staniforth and Wood (2003), the same system with mass coordinates was illustrated in Wood and Staniforth (2003), and a similar system is used in the UK Met Office (Davies et al. 2005). However, we have different considerations, which may not be fully provided for us in the literature, and thus we have to research and derive our own system. Instead of developing a totally new dynamics, the idea of incremental implementation is adopted. The incremental changes will be added into the existing GFS code, to minimize the amount of software development involved. The spectral transformation will be kept in the horizontal but the vertical discretization is changed. Since the deep-atmospheric nonhydrostatic dynamics are different from current hydrostatic system, a new discretization has to be done with appropriate conservation properties. The linearization of the equations for a semi-implicit time scheme in spectral space has to be constructed and all matrices related to linearized terms have to be redone.

In this note, the formulations of the deep-atmospheric nonhydrostatic dynamics in different vertical coordinates are presented in Section 2. The conservation properties in generalized vertical coordinates are illustrated in Section 3. The mass coordinates for the general concept of coordinate pressure from weight to determine a coordinate is introduced in Section 4. Based on this coordinate pressure, prognostic equations are given and discretization equations are obtained with all conservation in finite difference form in Section 5. The linearization for the semi-implicit scheme is illustrated in Section 6. The example of a semi-implicit semi-Lagrangian time scheme is given in Section 7, and a discussion of it is in Section 8. Several appendices are given for help with the detailed derivations and easy coding into existing models, including examples of the base state for linearization, linearized matrices, and initial condition preparation from existing hydrostatic states.

2. Deep atmospheric nonhydrostatic system on spherical coordinates

The three dimensional momentum equations for a deep atmospheric nonhydrostatic system on horizontal spherical coordinates and vertical height coordinates can be found in text books, such as Haltiner and Williams (1979), and can be written as

$$\frac{du}{dt}-\frac{uv\tan\phi}{r}+\frac{uw}{r}-(2\Omega\sin\phi)v+(2\Omega\cos\phi)w+\frac{1}{\rho}\left(\frac{1}{r\cos\phi}\frac{\partial p}{\partial\lambda}\right)=F_u \qquad (2.1a)$$

$$\frac{dv}{dt}+\frac{u^2\tan\phi}{r}+\frac{vw}{r}+(2\Omega\sin\phi)u+\frac{1}{\rho}\left(\frac{1}{r}\frac{\partial p}{\partial\phi}\right)=F_v \tag{2.1b}$$

$$\frac{dw}{dt}-\frac{u^2+v^2}{r}-(2\Omega\cos\phi)u+\frac{1}{\rho}\frac{\partial p}{\partial r}+g=F_w \tag{2.1c}$$

where all variables have the usual meaning, and the total derivative is

$$\frac{dA}{dt}=\frac{\partial A}{\partial t}+\frac{u}{r\cos\phi}\frac{\partial A}{\partial\lambda}+\frac{v}{r}\frac{\partial A}{\partial\phi}+w\frac{\partial A}{\partial r} \tag{2.2}$$

three-dimensional winds are

$$u=r\cos\phi\frac{d\lambda}{dt}$$
$$v=r\frac{d\phi}{dt} \tag{2.3}$$
$$w=\frac{dr}{dt}$$

where r is the distance from earth center, λ and ϕ are longitude and latitude, and g is a function of r as in

$$g=\bar{g}\frac{a^2}{r^2} \tag{2.4}$$

where $\bar{g}$ is mean gravitational force at a, which is mean surface height, and $r=a+z$. This deep atmospheric system can be simplified into a shallow atmospheric system by setting $r=a$ by Phillips (1966), which we used for current hydrostatic atmospheric models without considering a prognostic equation of vertical motion and its related Coriolis terms in horizontal momentum equations.

The related variables in the aforementioned momentum equations are pressure and density, and they are governed by the ideal gas law and continuity equation as

$$p=\sum_{i=1}^{N}p_i=\sum_{i=1}^{N}\rho_iR_iT=\rho\left(\sum_{i=1}^{N}\frac{\rho_iR_i}{\rho}\right)T=\rho\left(\sum_{i=1}^{N}q_iR_i\right)T=\rho RT \tag{2.5}$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{r\cos\phi}\left[\frac{\partial}{\partial\lambda}\langle\rho u\rangle+\frac{\partial}{\partial\phi}\langle\rho v\cos\phi\rangle\right]+\frac{1}{r^2}\frac{\partial}{\partial r}\langle\rho r^2w\rangle=F_\rho \tag{2.6}$$

where the sum of N individual constituent partial pressures results in a total pressure with a common temperature, the total density is a sum of densities of all constituents, and the total gas constant R is the sum of each constituent contribution weighted by the specific value. The continuity equation is suitable for each constituent and for the total densities. From the ideal gas law, we need a thermodynamic equation to govern temperature by an internal energy equation as

$$\frac{\partial\rho e}{\partial t}+\nabla\bullet\rho eV+p\nabla\bullet V=\rho Q \tag{2.7}$$

where e is the internal energy as in

$$e=\sum_{i=1}^{N}q_ie_i=\sum_{i=1}^{N}q_iC_{V_i}T=C_VT=\left(C_P-R\right)T=h-RT \tag{2.8}$$

Since we use the enthalpy h in our current NCEP GFS on a generalized hybrid coordinate, we will use enthalpy as a thermodynamic variable for consistency and backwards compatibility. After combining above two equations with the continuity equation, we have

$$\frac{dh}{dt}-\frac{\kappa h}{p}\frac{dp}{dt}=Q-\frac{h}{\rho}F_\rho \tag{2.9}$$

where $\kappa=\dfrac{R}{C_P}$, then we divide h into Eq. (2.9), and we have

$$\frac{d}{dt}\left(\ln\Theta\right)=\frac{Q}{h}-\frac{F_\rho}{\rho}-\ln\frac{p}{p_0}\frac{d\kappa}{dt} \tag{2.10}$$

where potential enthalpy

$$\Theta=\frac{h}{\pi}=\frac{h}{\left(p/p_0\right)^\kappa} \tag{2.11}$$

is conserved for an adiabatic system.

Then we apply a vertical coordinate conversion from the height coordinate to generalized vertical coordinates with all above prognostic variables, and we can group them as the following;

$$\frac{du}{dt}-\frac{uv\tan\phi}{r}+\frac{uw}{r}-f_s v+f_c w+\frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)=F_u \tag{2.12a}$$

$$\frac{dv}{dt}+\frac{u^2\tan\phi}{r}+\frac{vw}{r}+f_s u+\frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial\phi}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\phi}\right)=F_v \tag{2.12b}$$

$$\frac{dw}{dt}-\frac{u^2+v^2}{r}-f_c u+\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}+g=F_w \tag{2.12c}$$

$$\frac{dh}{dt}-\frac{\kappa h}{p}\frac{dp}{dt}=F_h \tag{2.12d}$$

$$\frac{\partial\beta}{\partial t}+\frac{\partial}{\partial\lambda}\left\langle\beta\dot{\lambda}\right\rangle+\frac{\partial}{\partial\phi}\left\langle\beta\dot{\phi}\right\rangle+\frac{\partial}{\partial\zeta}\left\langle\beta\dot{\zeta}\right\rangle=F_\rho^\beta \tag{2.12e}$$

$$\frac{dq_i}{dt}=F_{q_i} \tag{2.12f}$$

$$p=\rho\kappa h \tag{2.12g}$$

where

$$\frac{d()}{dt}=\frac{\partial()}{\partial t}+\dot{\lambda}\frac{\partial()}{\partial\lambda}+\dot{\phi}\frac{\partial()}{\partial\phi}+\dot{\zeta}\frac{\partial()}{\partial\zeta} \tag{2.13a}$$

$$\beta=\rho r^2\cos\phi\frac{\partial r}{\partial\zeta} \tag{2.13b}$$

$$w=\frac{\partial r}{\partial t}+\dot{\lambda}\frac{\partial r}{\partial\lambda}+\dot{\phi}\frac{\partial r}{\partial\phi}+\dot{\zeta}\frac{\partial r}{\partial\zeta} \tag{2.13c}$$

$$f_s=2\Omega\sin\phi \tag{2.13d}$$

$$f_c = 2\Omega\cos\phi \tag{2.13e}$$

The derivation to have β in the continuity equation can be found in Staniforth and Wood (2003), with details in Appendix A of this note. In their paper, they provided details of the derivation of all conservations: entropy, angular momentum, and total energy with mass conservation.

For programming our code, the equations have to change with spherical mapping by using virtual horizontal winds as

$$\frac{du^*}{dt} + \frac{u^* w}{r} - f_s v^* + f_c^* w + \frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) = F_u \tag{2.14a}$$

$$\frac{dv^*}{dt} + \frac{v^* w}{r} + f_s u^* + m^2 \frac{s^{*2}}{r}\sin\phi + \frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \varphi} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \varphi}\right) = F_v \tag{2.14b}$$

$$\frac{dw}{dt} - m^2 \frac{s^{*2}}{r} - m^2 f_c^* u^* + \frac{\kappa h}{p}\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r} + g = F_w \tag{2.14c}$$

$$\frac{dh}{dt} - \frac{\kappa h}{p}\frac{dp}{dt} = F_h \tag{2.14d}$$

$$\frac{\partial \rho^*}{\partial t} + m^2 \frac{\partial}{\partial \lambda}\left(\rho^* \frac{u^*}{r}\right) + m^2 \frac{\partial}{\partial \varphi}\left(\rho^* \frac{v^*}{r}\right) + \frac{\partial}{\partial \zeta}\left(\rho^* \dot{\zeta}\right) = F_\rho^* \tag{2.14e}$$

$$\frac{dq_i}{dt} = F_{q_i} \tag{2.14f}$$

$$p = \rho\kappa h \tag{2.14g}$$

where

$$\begin{aligned}\frac{d()}{dt} &= \frac{\partial()}{\partial t} + \frac{m^2 u^*}{r}\frac{\partial()}{\partial \lambda} + \frac{m^2 v^*}{r}\frac{\partial()}{\partial \varphi} + \dot{\zeta}\frac{\partial()}{\partial \zeta} \\ &= \frac{\partial()}{\partial t} + \dot{\lambda}\frac{\partial()}{\partial \lambda} + \dot{\varphi}\frac{\partial()}{\partial \varphi} + \dot{\zeta}\frac{\partial()}{\partial \zeta} \\ &= \frac{\partial()}{\partial t} + \dot{\lambda}\frac{\partial()}{\partial \lambda} + \dot{\phi}\frac{\partial()}{\partial \phi} + \dot{\zeta}\frac{\partial()}{\partial \zeta} \\ &= \frac{\partial()}{\partial t} + \dot{\lambda}\frac{\partial()}{\partial \lambda} + \dot{\mu}\frac{\partial()}{\partial \mu} + \dot{\zeta}\frac{\partial()}{\partial \zeta}\end{aligned} \tag{2.15a}$$

$$\Delta\varphi = m\Delta\phi = \frac{\Delta\phi}{\cos\phi} \tag{2.15b}$$

$$\Delta\mu = \frac{\Delta\phi}{m} = \cos\phi\Delta\phi \tag{2.15c}$$

$$m = \frac{1}{\cos\phi} \tag{2.15d}$$

$$u^* = u\cos\phi \tag{2.15e}$$

$$v^* = v\cos\phi \tag{2.15f}$$

$$s^{*2} = u^{*2} + v^{*2} \tag{2.15g}$$

$$\rho^* = \rho \frac{r^2}{a^2} \frac{\partial r}{\partial \zeta} \tag{2.15h}$$

$$f_s = 2\Omega \sin\phi \tag{2.15i}$$

$$f_c^* = 2\Omega \cos^2 \phi \tag{2.15j}$$

$$\kappa = \frac{R}{C_P} \tag{2.15k}$$

$$\gamma = \frac{C_P}{C_V} \tag{2.15l}$$

Again, the continuity equation in Eq. (2.14e) is obtained by detailed derivation shown in Appendix A.

3. Conservation properties of a deep atmospheric nonhydrostatic system

The deep atmospheric nonhydrostatic system on horizontal spherical coordinates and a generalized vertical coordinate in Eq. (2.12) has multiple conservation properties. The related details can be found in Staniforth and Wood (2003). For completeness, a comprehensive derivation is provided here.

First, we can see from the continuity equation (2.12e), that mass is conserved when the force term is zero. Doing a global total integration of Eq. (2.12e), we have

$$\oint_\delta \frac{\partial \beta}{\partial t} d\zeta\, d\lambda\, d\phi + \oint_\delta \frac{\partial \beta \dot{\lambda}}{\partial \lambda} d\zeta\, d\lambda\, d\phi + \oint_\delta \frac{\partial \beta \dot{\phi}}{\partial \phi} d\zeta\, d\lambda\, d\phi + \oint_\delta \frac{\partial \beta \dot{\zeta}}{\partial \zeta} d\zeta\, d\lambda\, d\phi = 0 \tag{3.1}$$

it can be further separated into horizontal and vertical integrals as

$$\begin{aligned} &\oint_s \left(\frac{\partial}{\partial t} \int_{\zeta_B}^{\zeta_T} \beta\, d\zeta + \frac{\partial}{\partial \lambda} \int_{\zeta_B}^{\zeta_T} \beta \dot{\lambda}\, d\zeta + \frac{\partial}{\partial \phi} \int_{\zeta_B}^{\zeta_T} \beta \dot{\phi}\, d\zeta \right) d\lambda\, d\phi \\ &- \oint_S \beta_T \left(\frac{\partial \zeta}{\partial t} + \dot{\lambda} \frac{\partial \zeta}{\partial \lambda} + \dot{\phi} \frac{\partial \zeta}{\partial \phi} - \dot{\zeta} \right)_T d\lambda\, d\phi + \oint_s \beta_B \left(\frac{\partial \zeta}{\partial t} + \dot{\lambda} \frac{\partial \zeta}{\partial \lambda} + \dot{\phi} \frac{\partial \zeta}{\partial \phi} - \dot{\zeta} \right)_B d\lambda\, d\phi = 0 \end{aligned} \tag{3.2}$$

where the last two terms are the top and bottom boundary conditions which are zero, and the second and third terms in the first group vanish in the horizontal total integral. Thus, finally, we have

$$\oiiint \frac{\partial}{\partial t} \beta\, d\zeta\, d\lambda\, d\phi = \frac{\partial}{\partial t} \oiiint \rho r^2 \cos\phi \frac{\partial r}{\partial \zeta} d\zeta\, d\lambda\, d\phi = \frac{\partial}{\partial t} \oint_V \rho\, dv = 0 \tag{3.3}$$

which indicates a conservation of total mass. Using the same step we can have entropy conservation by combining the following potential enthalpy equations

$$\frac{d\Theta}{dt} = \frac{\partial \Theta}{\partial t} + \dot{\lambda} \frac{\partial \Theta}{\partial \lambda} + \dot{\phi} \frac{\partial \Theta}{\partial \phi} + \dot{\zeta} \frac{\partial \Theta}{\partial \zeta} = 0 \tag{3.4}$$

with the continuity equation, Eq. (2.12e), to be

$$\frac{\partial}{\partial t}(\beta\Theta)+\frac{\partial}{\partial\lambda}\left(\dot{\lambda}\,\beta\Theta\right)+\frac{\partial}{\partial\phi}\left(\dot{\phi}\,\beta\Theta\right)+\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\,\beta\Theta\right)=0 \tag{3.5}$$

For the total global integral of the above, we have

$$\iiint\frac{\partial}{\partial t}\langle\beta\Theta\rangle d\zeta\,d\lambda\,d\phi=\frac{\partial}{\partial t}\iiint\beta\Theta\,d\zeta\,d\lambda\,d\phi=\frac{\partial}{\partial t}\iiint\rho\Theta\,dv=0 \tag{3.6}$$

which indicates conservation of total mass weighted potential enthalpy.

Next, let's check the behavior of the total integral of angular momentum. The angular momentum per unit mass can be defined as

$$A=r\cos\phi\left(u+\Omega r\cos\phi\right) \tag{3.7}$$

So the total derivative of angular momentum is

$$\begin{aligned}\frac{dA}{dt}&=r\cos\phi\frac{du}{dt}+\left(u+2\Omega r\cos\phi\right)\frac{d}{dt}\langle r\cos\phi\rangle\\&=r\cos\phi\frac{du}{dt}+\left(u+2\Omega r\cos\phi\right)\left(w\cos\phi-v\sin\phi\right)\end{aligned} \tag{3.8}$$

puting the u momentum equation, Eq. (2.12a), into the above equation, we have

$$\frac{dA}{dt}=r\cos\phi\left[F_u-\frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)\right] \tag{3.9}$$

Again, combining with the continuity equation, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}(\beta A)+\frac{\partial}{\partial\lambda}\left(\frac{u\beta A}{r\cos\phi}\right)+\frac{\partial}{\partial\phi}\left(\frac{v\beta A}{r}\right)+\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\,\beta A\right)\\&=\beta r\cos\phi\left[F_u-\frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)\right]\\&=-r^2\cos\phi\frac{\partial r}{\partial\zeta}\frac{\partial p}{\partial\lambda}+r^2\cos\phi\frac{\partial p}{\partial\zeta}\frac{\partial r}{\partial\lambda}+\beta r\cos\phi F_u\\&=-\frac{\partial}{\partial\lambda}\left\langle pr^2\cos\phi\frac{\partial r}{\partial\zeta}\right\rangle+p\frac{\partial}{\partial\lambda}\left\langle r^2\cos\phi\frac{\partial r}{\partial\zeta}\right\rangle\\&+\frac{\partial}{\partial\zeta}\left\langle pr^2\cos\phi\frac{\partial r}{\partial\lambda}\right\rangle-p\frac{\partial}{\partial\zeta}\left\langle r^2\cos\phi\frac{\partial r}{\partial\lambda}\right\rangle+\beta r\cos\phi F_u\end{aligned} \tag{3.10}$$

where the second and fourth terms at right-hand-side (RHS) cancel each other due to

$$\frac{\partial r^2\frac{\partial r}{\partial\zeta}}{\partial\lambda}=\frac{1}{3}\frac{\partial\frac{\partial r^3}{\partial\zeta}}{\partial\lambda}=\frac{1}{3}\frac{\partial\frac{\partial r^3}{\partial\lambda}}{\partial\zeta}=\frac{\partial r^2\frac{\partial r}{\partial\lambda}}{\partial\zeta} \tag{3.10a}$$

Thus, for the total integral of Eq. (3.10) with the above cancellation, we have

$$\frac{\partial}{\partial t}\oiiint\rho A\,dv=\oiiint\rho r\cos\phi F_u\,dv+\oiint\left(p\frac{\partial r}{\partial\lambda}\right)_T ds-\oiint\left(p\frac{\partial r}{\partial\lambda}\right)_B ds \tag{3.11}$$

which is zero under the condition of no source term, zero pressure at top of atmosphere, and no ground surface gradient. Thus it indicates that total integration of mass weighted angular momentum is governed by the ground surface torques.

Next, let's multiply Eq. (2.12a) by u, Eq. (2.12b) by v, and Eq. (2.12c) by w, and add them together, so we have the kinetic energy equation as

$$\begin{aligned}&\frac{\partial K}{\partial t}+\dot{\lambda}\frac{\partial K}{\partial \lambda}+\dot{\phi}\frac{\partial K}{\partial \phi}+\dot{\zeta}\frac{\partial K}{\partial \zeta}\\&=-\frac{1}{\rho}\dot{\lambda}\left(\frac{\partial p}{\partial \lambda}-\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right)-\frac{1}{\rho}\dot{\phi}\left(\frac{\partial p}{\partial \phi}-\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \phi}\right)-\frac{1}{\rho}w\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}-gw\end{aligned} \tag{3.12}$$

where $K=\frac{1}{2}\left(u^2+v^2+w^2\right)$ as kinetic energy. Combining it with the continuity equation we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}(\beta K)+\frac{\partial}{\partial \lambda}\left(\dot{\lambda}\beta\left(K+\frac{p}{\rho}\right)\right)+\frac{\partial}{\partial \phi}\left(\dot{\phi}\beta\left(K+\frac{p}{\rho}\right)\right)+\frac{\partial}{\partial \zeta}\left(\dot{\zeta}\beta\left(K+\frac{p}{\rho}\right)\right)\\&=p\frac{\partial}{\partial \lambda}\left(\frac{\beta}{\rho}\dot{\lambda}\right)+p\frac{\partial}{\partial \phi}\left(\frac{\beta}{\rho}\dot{\phi}\right)+p\frac{\partial}{\partial \zeta}\left(\frac{\beta}{\rho}\dot{\zeta}\right)-\frac{\partial}{\partial \zeta}\left(p\frac{\beta}{\rho}\frac{\partial r}{\partial t}\frac{\partial \zeta}{\partial r}\right)+p\frac{\partial}{\partial \zeta}\left(\frac{\beta}{\rho}\frac{\partial r}{\partial t}\frac{\partial \zeta}{\partial r}\right)-\beta gw\\&=p\left(\frac{\partial}{\partial \lambda}\left(\frac{\beta}{\rho}\dot{\lambda}\right)+\frac{\partial}{\partial \phi}\left(\frac{\beta}{\rho}\dot{\phi}\right)+\frac{\partial}{\partial \zeta}\left(\frac{\beta}{\rho}\dot{\zeta}\right)\right)-\frac{\partial}{\partial \zeta}\left(pr^2\cos\phi\frac{\partial r}{\partial t}\right)+p\frac{\partial}{\partial \zeta}\left(r^2\cos\phi\frac{\partial r}{\partial t}\right)-\beta gw\\&=p\left(\frac{\partial}{\partial t}\left\langle\frac{\beta}{\rho}\right\rangle+\frac{\partial}{\partial \lambda}\left\langle\frac{\beta}{\rho}\dot{\lambda}\right\rangle+\frac{\partial}{\partial \phi}\left\langle\frac{\beta}{\rho}\dot{\phi}\right\rangle+\frac{\partial}{\partial \zeta}\left\langle\frac{\beta}{\rho}\dot{\zeta}\right\rangle\right)-\frac{\partial}{\partial \zeta}\left\langle pr^2\cos\phi\frac{\partial r}{\partial t}\right\rangle-\beta gw\end{aligned} \tag{3.13}$$

For potential energy, we start from

$$gw=\frac{d\Phi}{dr}\frac{dr}{dt}=\frac{\partial \Phi}{\partial t}+\dot{\lambda}\frac{\partial \Phi}{\partial \lambda}+\dot{\phi}\frac{\partial \Phi}{\partial \phi}+\dot{\zeta}\frac{\partial \Phi}{\partial \zeta} \tag{3.14}$$

then combining with the continuity equation, we obtain

$$\frac{\partial}{\partial t}\langle\beta\Phi\rangle+\frac{\partial}{\partial \lambda}\left\langle\dot{\lambda}\beta\Phi\right\rangle+\frac{\partial}{\partial \phi}\left\langle\dot{\phi}\beta\Phi\right\rangle+\frac{\partial}{\partial \zeta}\left\langle\dot{\zeta}\beta\Phi\right\rangle=\beta gw \tag{3.15}$$

Last, for thermodynamic energy, we start from the thermodynamic equation with the following variation as

$$\frac{d}{dt}\langle C_P T\rangle-\frac{1}{\rho}\frac{dp}{dt}=\frac{d}{dt}\langle C_V T\rangle+\frac{d}{dt}\langle RT\rangle-\frac{1}{\rho}\frac{dp}{dt}=\frac{d}{dt}\langle C_V T\rangle+p\frac{d}{dt}\left\langle\frac{1}{\rho}\right\rangle=0 \tag{3.16}$$

Again, combining with the continuity equation, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\langle\beta C_V T\rangle+\frac{\partial}{\partial \lambda}\left\langle\dot{\lambda}\beta C_V T\right\rangle+\frac{\partial}{\partial \phi}\left\langle\dot{\phi}\beta C_V T\right\rangle+\frac{\partial}{\partial \zeta}\left\langle\dot{\zeta}\beta C_V T\right\rangle\\&=-p\left[\frac{\partial}{\partial t}\left(\frac{\beta}{\rho}\right)+\frac{\partial}{\partial \lambda}\left(\dot{\lambda}\frac{\beta}{\rho}\right)+\frac{\partial}{\partial \phi}\left(\dot{\phi}\frac{\beta}{\rho}\right)+\frac{\partial}{\partial \zeta}\left(\dot{\zeta}\frac{\beta}{\rho}\right)\right]\end{aligned} \tag{3.17}$$

Summing Eqs. (3.13), (3.15) and (3.17), and integrating globally, we obtain

$$\frac{\partial}{\partial t}\oiiint \beta\left(K+\Phi+C_V T\right)d\lambda\, d\phi\, d\zeta=\frac{\partial}{\partial t}\oiiint \rho\left(K+\Phi+C_V T\right)dv=0 \tag{3.18}$$

with a top boundary condition of zero pressure and the bottom boundary condition of zero local change of terrain at the ground surface. As mentioned previously, we paraphrase all conservation properties here in this note, otherwise, you can find a detailed derivation in Staniforth and Wood (2003). The most tricky or tedious derivation is to convert the density equation in z to a generalized vertical coordinate, in which we use ρ in z coordinate but $\beta = \rho r^2 \cos\phi \dfrac{\partial r}{\partial \zeta}$ for the continuity equation in generalized coordinates, and $\rho^* = \rho \dfrac{r^2}{a^2}\dfrac{\partial r}{\partial \zeta}$ for spherical mapping and the generalized vertical coordinate. The detailed derivation of these three conversions in the density equation can be found in Appendix A.

4. Mass coordinates by coordinate pressure

Before we do a discretization, we would like to do incremental changes to the vertical grid, which is similar to our existing grid system in Fig. 1. All the indices are integers to make them easy to follow and code into a model. The variables at model levels are noted with a hat, and variables without a hat are in model layers.

Continuity is used for constructing mass coordinates for most mass based vertical coordinates. There are several ways to make mass coordinates, such as Laprise (1992), Juang (1992, 2000) for nonhydrostatic systems, and Staniforth and Wood (2003) and Wood and Staniforth (2003) for deep atmosphere nonhydrostatic systems. Again, considering incremental changes, we will give a concept to construct mass coordinates similar to a hydrostatic system in the continuity equation. The general concept of measured mean pressure at any given location in the vertical sense is the weight on top of the location divided by the area at the given location. Let's give a mass above any given level as

$$Mass = \int_{\phi_1}^{\phi_2}\int_{\lambda_1}^{\lambda_2}\int_{\zeta}^{\zeta_{TOP}} \rho r^2 \cos\phi \frac{\partial r}{\partial \zeta} d\zeta\, d\lambda\, d\phi \tag{4.1}$$

then project this mass on earth radius surface as the following

$$\frac{Mass\ \bar{g}}{a^2 \int_{\phi_1}^{\phi_2}\int_{\lambda_1}^{\lambda_2} \cos\phi\, d\lambda\, d\phi} = \frac{\int_{\phi_1}^{\phi_2}\int_{\lambda_1}^{\lambda_2}\int_{\zeta}^{\zeta_{TOP}} \rho \bar{g} \frac{r^2}{a^2}\frac{\partial r}{\partial \zeta} d\zeta \cos\phi\, d\lambda\, d\phi}{\int_{\phi_1}^{\phi_2}\int_{\lambda_1}^{\lambda_2} \cos\phi\, d\lambda\, d\phi} \tag{4.2}$$

which can be deduced to be a pressure-alike variable called a coordinate pressure as

$$\tilde{\bar{p}}_{\zeta} = \int_{\zeta}^{\zeta_{TOP}} \rho \bar{g} \frac{r^2}{a^2}\frac{\partial r}{\partial \zeta} d\zeta \tag{4.3}$$

Then let the coordinate pressure at top be zero, we have

$$\tilde{\bar{p}}_{\zeta} = -\int_{\zeta}^{\zeta_{TOP}} \frac{\partial \tilde{\bar{p}}}{\partial \zeta} d\zeta \tag{4.4}$$

thus, we get the following coordinate relationship from Eqs. (4.3) and (4.4) as

$$\frac{\partial \tilde{\bar{p}}}{\partial \zeta} = -\rho \bar{g} \frac{r^2}{a^2}\frac{\partial r}{\partial \zeta} = -\rho^* \bar{g} \tag{4.5}$$

which is similar to the hydrostatic relationship in a hydrostatic system.

From our current generalized hybrid coordinate system for the continuity equation, Eq. (4.5) has a form similar to the hydrostatic equation as mentioned. In this way, the density in Eq. (2.14e) can be replaced by coordinate pressure gradient, which defines the density with a constant gravitational force as

$$\frac{\partial}{\partial t}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\right)+m^2\left(\frac{\partial}{\partial\lambda}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{u^*}{r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{v^*}{r}\right)\right)+\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\dot{\zeta}\right)=0 \tag{4.6}$$

and it can be further discretized as the following form

$$\frac{\partial\Delta\tilde{\bar{p}}_k}{\partial t}+m^2\left(\frac{\partial}{\partial\lambda}\left(\frac{u^*\Delta\tilde{\bar{p}}}{r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{v^*\Delta\tilde{\bar{p}}}{r}\right)\right)_k+\left(\dot{\zeta}\,\widehat{\frac{\partial\tilde{\bar{p}}}{\partial\zeta}}\right)_k-\left(\dot{\zeta}\,\widehat{\frac{\partial\tilde{\bar{p}}}{\partial\zeta}}\right)_{k+1}=0 \tag{4.7}$$

where $\Delta\tilde{\bar{p}}_k=\hat{\tilde{\bar{p}}}_k-\hat{\tilde{\bar{p}}}_{k+1}$, and summed from top of model to any given layer as

$$\frac{\partial\hat{\tilde{\bar{p}}}_k}{\partial t}+\sum_{i=k}^{K}m^2\left(\frac{\partial}{\partial\lambda}\left(\frac{u_i^*\Delta\tilde{\bar{p}}_i}{r_i}\right)+\frac{\partial}{\partial\varphi}\left(\frac{v_i^*\Delta\tilde{\bar{p}}_i}{r_i}\right)\right)+\left(\widehat{\frac{\partial\tilde{\bar{p}}}{\partial\zeta}}\dot{\zeta}\right)_k=0 \tag{4.8}$$

or totally summed to the ground as

$$\frac{\partial\hat{\tilde{\bar{p}}}_s}{\partial t}+\sum_{i=1}^{K}m^2\left(\frac{\partial}{\partial\lambda}\left(\frac{u_i^*\Delta\tilde{\bar{p}}_i}{r_i}\right)+\frac{\partial}{\partial\varphi}\left(\frac{v_i^*\Delta\tilde{\bar{p}}_i}{r_i}\right)\right)=0 \tag{4.9}$$

or in total derivative form for a semi-Lagrangian scheme as

$$\frac{d_h}{d_h t}\Delta\tilde{\bar{p}}_k+m^2\Delta\tilde{\bar{p}}_k\left(\frac{\partial}{\partial\lambda}\left(\frac{u^*}{r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{v^*}{r}\right)\right)_k+\left(\dot{\zeta}\,\widehat{\frac{\partial\tilde{\bar{p}}}{\partial\zeta}}\right)_k-\left(\dot{\zeta}\,\widehat{\frac{\partial\tilde{\bar{p}}}{\partial\zeta}}\right)_{k+1}=0 \tag{4.10}$$

and summed from top of model to the ground as

$$\frac{D\hat{\tilde{\bar{p}}}_s}{Dt}+\sum_{i=1}^{K}m^2\Delta\tilde{\bar{p}}_i\left(\frac{\partial}{\partial\lambda}\left(\frac{u_i^*}{r_i}\right)+\frac{\partial}{\partial\varphi}\left(\frac{v_i^*}{r_i}\right)\right)=0 \tag{4.11}$$

where

$$\frac{d_h}{d_h t}\left(\Delta\tilde{\bar{p}}_k\right)=\frac{\partial}{\partial t}\left(\Delta\tilde{\bar{p}}_k\right)+m^2\left(\frac{u^*}{r}\frac{\partial\Delta\tilde{\bar{p}}}{\partial\lambda}+\frac{v^*}{r}\frac{\partial\Delta\tilde{\bar{p}}}{\partial\varphi}\right)_k \tag{4.12}$$

and

$$\frac{D\hat{\tilde{\bar{p}}}_s}{Dt}=\frac{\partial\hat{\tilde{\bar{p}}}_s}{\partial t}+\sum_{i=1}^{K}m^2\left(\frac{u_i^*}{r_i}\frac{\partial}{\partial\lambda}+\frac{v_i^*}{r_i}\frac{\partial}{\partial\varphi}\right)\Delta\tilde{\bar{p}}_i \tag{4.13}$$

Let's return to the coordinate pressure definition and replace density with the ideal gas relationship, then we have the following form for a vertical coordinate relationship with coordinate pressure definition as

$$\frac{\partial r}{\partial\zeta}=-\frac{\kappa h}{p\bar{g}}\frac{a^2}{r^2}\frac{\partial\tilde{\bar{p}}}{\partial\zeta} \tag{4.14}$$

which can be used to replace all coordinate terms related to the pressure gradients of momentum equations. Put all *r*'s into left hand side and apply derivative then do a discretization so we have a relationship between coordinate pressure and height as

$$\hat{r}_{k+1}^3 = \hat{r}_k^3 + 3\left(\frac{\kappa h}{p\bar{g}}\right)_k a^2\left(\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}\right) \tag{4.15}$$

This becomes the diagnostic equation for height. Going back to Fig. 1, all prognostic variables are in model layers, such as the three dimensional momentum, enthalpy, and pressure. And coordinate pressure and model level height are defined by the continuity equation and mass coordinate definition here. The constraint relationship to use on model layer heights from model levels in Eq. (4.15) has to be determined by the conservation requirement as in the following section.

5. Vertical discretization based on multiple conservations

In this section, we will use the mass conservation through coordinate pressure discussed in the previous section to do further discretization of all equations based on multi-conserving properties of angular momentum, total energy, and entropy (potential enthalpy) .

5.1 Angular momentum considerations

First, we start from angular momentum, which is given in Section 3, Eq. (3.7), and its total derivative without the source term is given as

$$\frac{dA}{dt} = -\frac{\kappa h}{p}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) \tag{5.1}$$

where we use *A* as its definition with u, not u* in Eq. (3.8), and we expand the total derivative with m in the advection term as

$$\frac{\partial A}{\partial t} + m^2\left(\frac{u^*}{r}\frac{\partial A}{\partial \lambda} + \frac{v^*}{r}\frac{\partial A}{\partial \varphi}\right) + \dot{\zeta}\frac{\partial A}{\partial \zeta} = -\frac{\kappa h}{p}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) \tag{5.2}$$

combining it with the continuity equation in the previous section's Eq. (4.6), we have

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial \tilde{\bar{p}}}{\partial \zeta}A\right) + m^2\left(\frac{\partial}{\partial \lambda}\left(\frac{\partial \tilde{\bar{p}}}{\partial \zeta}\frac{u^*}{r}A\right) + \frac{\partial}{\partial \varphi}\left(\frac{\partial \tilde{\bar{p}}}{\partial \zeta}\frac{v^*}{r}A\right)\right) + \frac{\partial}{\partial \zeta}\left(\frac{\partial \tilde{\bar{p}}}{\partial \zeta}\dot{\zeta}A\right) \\ &= -\frac{\partial \tilde{\bar{p}}}{\partial \zeta}\frac{\kappa h}{p}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) \end{aligned} \tag{5.3}$$

After globally integrating the above equation, we will obtain a global mass weighted angular momentum change. In other words, left-hand-side (LHS) of Eq. (5.3) retains only the local change and RHS of Eq. (5.3) retains only the ground surface pressure gradient with surface height. Thus, after vertically integrating the RHS of Eq. (5.3) with coordinate pressure definition, we have

$$\int_{\zeta_S}^{\zeta_T}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\left(\frac{\partial p}{\partial\lambda}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)d\zeta=\int_{\zeta_S}^{\zeta_T}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\lambda}-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)d\zeta$$
$$=\int_{\zeta_S}^{\zeta_T}\left(-\frac{\partial r}{\partial\zeta}\frac{\bar{g}r^2}{a^2}\frac{\partial p}{\partial\lambda}+\frac{\bar{g}r^2}{a^2}\frac{\partial p}{\partial\zeta}\frac{\partial r}{\partial\lambda}\right)d\zeta \tag{5.4}$$
$$=\int_{\zeta_S}^{\zeta_T}\frac{\bar{g}}{3a^2}\left(-\frac{\partial r^3}{\partial\zeta}\frac{\partial p}{\partial\lambda}+\frac{\partial p}{\partial\zeta}\frac{\partial r^3}{\partial\lambda}\right)d\zeta$$

then we apply a derivative by chain rule, and alter the sequence of derivatives to a single variable, and the right hand side becomes

$$\int_{\zeta_S}^{\zeta_T}\frac{\bar{g}}{3a^2}\left(-\frac{\partial r^3}{\partial\zeta}\frac{\partial p}{\partial\lambda}+\frac{\partial}{\partial\lambda}\left(r^3\frac{\partial p}{\partial\zeta}\right)-r^3\frac{\partial}{\partial\lambda}\left(\frac{\partial p}{\partial\zeta}\right)\right)d\zeta=$$
$$\frac{\partial}{\partial\lambda}\left(\int_{\zeta_S}^{\zeta_T}\frac{\bar{g}}{3a^2}r^3\frac{\partial p}{\partial\zeta}d\zeta\right)-\frac{\bar{g}r_T^3}{3a^2}\frac{\partial p_T}{\partial\lambda}+\frac{\bar{g}r_S^3}{3a^2}\frac{\partial p_S}{\partial\lambda} \tag{5.5}$$

When globally integrating of LHS and RHS in the above equation with pressure at the top level either constant or zero, we have

$$\sum_{k=1}^{K}\left[\left(\hat{r}_{k+1}^3-\hat{r}_k^3\right)\frac{\partial p_k}{\partial\lambda}+r_k^3\left(\frac{\partial\hat{p}_{k+1}}{\partial\lambda}-\frac{\partial\hat{p}_k}{\partial\lambda}\right)\right]=-\hat{r}_S^3\frac{\partial p_S}{\partial\lambda} \tag{5.6}$$

let $p_k=f\left(\hat{p}_{k+1},\hat{p}_k\right)$, so $p_k=\frac{\partial f_k}{\partial\hat{p}_{k+1}}\hat{p}_{k+1}+\frac{\partial f_k}{\partial\hat{p}_k}\hat{p}_k$ and $\frac{\partial f_k}{\partial\hat{p}_{k+1}}+\frac{\partial f_k}{\partial\hat{p}_k}=1$. Put these relations into above equation and expand it as

$$\left\langle\left(\hat{r}_{K+1}^3-\hat{r}_K^3\right)\frac{\partial f_K}{\partial\hat{p}_{K+1}}+r_K^3\right\rangle\frac{\partial\hat{p}_{K+1}}{\partial\lambda}+\left\langle\left(\hat{r}_{K+1}^3-\hat{r}_K^3\right)\frac{\partial f_K}{\partial\hat{p}_K}-r_K^3\right\rangle\frac{\partial\hat{p}_K}{\partial\lambda}$$
$$+\left\langle\left(\hat{r}_K^3-\hat{r}_{K-1}^3\right)\frac{\partial f_{K-1}}{\partial\hat{p}_K}+r_{K-1}^3\right\rangle\frac{\partial\hat{p}_K}{\partial\lambda}+\left\langle\left(\hat{r}_K^3-\hat{r}_{K-1}^3\right)\frac{\partial f_{K-1}}{\partial\hat{p}_{K-1}}-r_{K-1}^3\right\rangle\frac{\partial\hat{p}_{K-1}}{\partial\lambda}+\ldots\ldots \tag{5.7}$$
$$+\left\langle\left(\hat{r}_2^3-\hat{r}_1^3\right)\frac{\partial f_1}{\partial\hat{p}_2}+r_1^3\right\rangle\frac{\partial\hat{p}_2}{\partial\lambda}+\left\langle\left(\hat{r}_2^3-\hat{r}_1^3\right)\frac{\partial f_1}{\partial\hat{p}_1}-r_1^3\right\rangle\frac{\partial\hat{p}_1}{\partial\lambda}=-\hat{r}_S^3\frac{\partial p_S}{\partial\lambda}=-\hat{r}_1^3\frac{\partial\hat{p}_1}{\partial\lambda}$$

This will be always true if each group of coefficients at the same level of pressure derivative is zero. Thus, we get

$$\left\langle\left(\hat{r}_{k+1}^3-\hat{r}_k^3\right)\frac{\partial f_k}{\partial\hat{p}_k}-r_k^3\right\rangle+\left\langle\left(\hat{r}_k^3-\hat{r}_{k-1}^3\right)\frac{\partial f_{k-1}}{\partial\hat{p}_k}+r_{k-1}^3\right\rangle=0 \tag{5.8}$$

$$\left\langle\left(\hat{r}_2^3-\hat{r}_1^3\right)\frac{\partial f_1}{\partial\hat{p}_1}-r_1^3\right\rangle=-\hat{r}_1^3 \tag{5.9}$$

with pressure at the top level as a zero derivative.

For simplicity we can let $\frac{\partial f_k}{\partial \hat{p}_{k+1}} = \frac{\partial f_k}{\partial \hat{p}_k} = \frac{1}{2}$ as in the hydrostatic system, so p at a model layer is the mean of its neighboring pressures at model levels. Then we get relationship for heights between layer and its neighboring levels as

$$r_k^3 = \frac{1}{2}\hat{r}_{k+1}^3 + \frac{1}{2}\hat{r}_k^3 \tag{5.10}$$

Based on this relationship and the relationship for heights at model levels Eq. (4.15), we can use another way to describe the relationship between height at model levels and model layers as

$$r_k^3 = \hat{r}_k^3 + \frac{3a^2}{2\overline{g}}\left(\frac{\kappa h}{p}\right)_k\left(\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}\right) \tag{5.11}$$

We can use Eq. (4.15) to have model level height as a summation from ground level up to any given model level as

$$\hat{r}_k^3 = \hat{r}_1^3 + 3\frac{a^2}{\overline{g}}\sum_{i=1}^{k-1}\left[\kappa_i h_i \frac{\hat{\overline{\overline{p}}}_i - \hat{\overline{\overline{p}}}_{i+1}}{p_i}\right] \tag{5.12}$$

and the relationships in Eqs. (5.10) and (5.11) have model layer height as the summation from ground level up to any given model layer as

$$r_k^3 = \hat{r}_1^3 + 3\frac{a^2}{\overline{g}}\sum_{i=1}^{k-1}\left[\kappa_i h_i \frac{\hat{\overline{\overline{p}}}_i - \hat{\overline{\overline{p}}}_{i+1}}{p_i}\right] + \frac{3}{2}\frac{a^2}{\overline{g}}\kappa_k h_k \frac{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}{p_k} \tag{5.13}$$

Thus, the pressure gradient in the latitudinal momentum equation can be written as

$$\begin{aligned}\left[\frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right)\right]_k &= \frac{\kappa_k h_k}{p_k}\frac{a}{r_k}\frac{\partial p_k}{a\partial \lambda} + \frac{r_k^2}{a^2}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}\frac{a}{r_k}\frac{\overline{g}\partial r_k}{a\partial \lambda} \\ &= \frac{\kappa_k h_k}{p_k}\frac{a}{r_k}\frac{\partial p_k}{a\partial \lambda} + \frac{1}{3a^2}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}\frac{a}{r_k}\frac{\overline{g}\partial r_k^3}{a\partial \lambda}\end{aligned} \tag{5.14a}$$

then we put Eq. (5.13) into last terms of Eq. (5.14a), we have

$$\begin{aligned}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}\frac{\overline{g}}{3a^2}\frac{a}{r_k}\frac{\partial r_k^3}{a\partial \lambda} = \frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}\Bigg\{&\frac{\overline{g}}{3a^2}\frac{a}{r_k}\frac{\partial \hat{r}_1^3}{a\partial \lambda} \\ &+ \frac{a}{r_k}\left\langle \frac{1}{2}\frac{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}{p_k}\frac{\partial}{a\partial \lambda}\left(\kappa_k h_k\right) + \sum_{i=1}^{k-1}\frac{\hat{\overline{\overline{p}}}_i - \hat{\overline{\overline{p}}}_{i+1}}{p_i}\frac{\partial}{a\partial \lambda}\left(\kappa_i h_i\right)\right\rangle \\ &+ \frac{a}{r_k}\left\langle \frac{1}{2}\frac{\kappa_k h_k}{p_k}\frac{\partial}{a\partial \lambda}\left(\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}\right) + \sum_{i=1}^{k-1}\frac{\kappa_i h_i}{p_i}\frac{\partial}{a\partial \lambda}\left(\hat{\overline{\overline{p}}}_i - \hat{\overline{\overline{p}}}_{i+1}\right)\right\rangle \\ &- \frac{a}{r_k}\left\langle \kappa_k h_k \frac{1}{2}\frac{\hat{\overline{\overline{p}}}_k - \hat{\overline{\overline{p}}}_{k+1}}{p_k^2}\frac{\partial p_k}{a\partial \lambda} + \sum_{i=1}^{k-1}\kappa_i h_i \frac{\hat{\overline{\overline{p}}}_i - \hat{\overline{\overline{p}}}_{i+1}}{p_i^2}\frac{\partial p_i}{a\partial \lambda}\right\rangle\Bigg\}\end{aligned} \tag{5.14}$$

where λ can be replaced by φ for longitudinal momentum, and

$$
\begin{aligned}
\frac{\partial \kappa h}{\partial \lambda} &= \kappa \frac{\partial h}{\partial \lambda} + h\frac{\partial \kappa}{\partial \lambda} \\
&= \frac{R}{C_P}\frac{\partial h}{\partial \lambda} + h\frac{C_P\dfrac{\partial R}{\partial \lambda} - R\dfrac{\partial C_P}{\partial \lambda}}{C_P^2} \\
&= \frac{R}{C_P}\frac{\partial h}{\partial \lambda} + h\frac{C_P\sum_{i=1}^{N} R_i\dfrac{\partial q_i}{\partial \lambda} - R\sum_{i=1}^{N} C_{Pi}\dfrac{\partial q_i}{\partial \lambda}}{C_P^2}
\end{aligned}
\tag{5.15}
$$

In the case of a semi-Lagrangian model with tracers only in grid point space, the derivatives of specific tracers are not available. To save on spectral transforms without spectral transformation for N tracers, we will compute r (height) at a model layer by Eqs. (5.10) and (5.12), doing spectral derivatives in r for the momentum equations as follows

$$
\frac{du_k^*}{dt} = -\frac{u_k^* w_k}{r_k} + f_s v_k^* - f_c^* w_k - \frac{\kappa_k h_k}{p_k}\frac{a}{r_k}\frac{\partial p_k}{a\partial \lambda} - \frac{\bar{g}}{3a^2}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\tilde{\tilde{p}}}_k - \hat{\tilde{\tilde{p}}}_{k+1}}\frac{a}{r_k}\frac{\partial r_k^3}{a\partial \lambda} \tag{5.16a}
$$

$$
\frac{dv_k^*}{dt} = -\frac{v_k^* w_k}{r_k} - f_s u_k^* - m^2\frac{s_k^{*2}}{r_k}\sin\phi - \frac{\kappa_k h_k}{p_k}\frac{a}{r_k}\frac{\partial p_k}{a\partial \varphi} - \frac{\bar{g}}{3a^2}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\tilde{\tilde{p}}}_k - \hat{\tilde{\tilde{p}}}_{k+1}}\frac{a}{r_k}\frac{\partial r_k^3}{a\partial \varphi} \tag{5.16b}
$$

5.2 Total energy conservation

Next, let's check into total energy conservation. From Section 3, we know that the energy conversion term exists in the kinetic energy equation as well as the thermodynamic equation. And we concluded that conservation is valid while the energy conversion term used in the kinetic energy equation applies to the thermodynamic energy equation. This is true in the hydrostatic system shown in Juang (2005) and (2011) except the conservation in the deep atmospheric nonhydrostatic system required three-dimensional advection of the pressure gradient instead of only horizontal advection, as

$$
\begin{aligned}
&-\frac{\partial \tilde{\bar{p}}}{\partial \zeta}\left[m^2 u^* \frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) + m^2 v^* \frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \varphi} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \varphi}\right) + w\frac{\kappa h}{p}\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\right] \\
&= -\frac{\partial \tilde{\bar{p}}}{\partial \zeta}\frac{\kappa h}{p}\dot{\lambda}\left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) - \frac{\partial \tilde{\bar{p}}}{\partial \zeta}\frac{\kappa h}{p}\dot{\mu}\left(\frac{\partial p}{\partial \mu} - \frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \mu}\right) - \frac{\partial \tilde{\bar{p}}}{\partial \zeta}w\frac{\kappa h}{p}\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}
\end{aligned}
\tag{5.17}
$$

We use μ here, as an easy way to absorb mapping factor m into the operators, so we can use the following local symbolic derivative definitions to simplify the derivation later in this subsection as

$$
\begin{aligned}
&\dot{\lambda}\frac{\partial A}{\partial \lambda} + \dot{\mu}\frac{\partial A}{\partial \mu} = V_H \bullet \nabla A \\
&\frac{\partial \dot{\lambda} A}{\partial \lambda} + \frac{\partial \dot{\mu} A}{\partial \mu} = \nabla \bullet (V_H A)
\end{aligned}
\tag{5.18}
$$

Don't confuse this local gradient here from the other definition.

Thus, Eq. (5.17) can be rewritten, performing the chain rule, and expanding w by the total derivative of r as follows

$$
\begin{aligned}
&-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\bullet\nabla_H p+\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}V_H\bullet\nabla_H r-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}w\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\\
&=-\nabla_H\bullet\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)+p\nabla_H\bullet\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)\\
&+\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}V_H\bullet\nabla_H r-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\left(\frac{\partial r}{\partial t}+V_H\bullet\nabla_H r+\dot{\zeta}\frac{\partial r}{\partial\zeta}\right)\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}
\end{aligned}
\tag{5.19a}
$$

then the horizontal advection of r in the last two terms will be cancelled out with the remaining

$$
=-\nabla_H\bullet\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)+p\nabla_H\bullet\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial t}-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\dot{\zeta}
\tag{5.19b}
$$

The first term will be zero after global integral, and the last two terms can be expanded by the chain rule for vertical derivatives on p as

$$
\begin{aligned}
&-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial t}-\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\dot{\zeta}\\
&=-\frac{\partial}{\partial\zeta}\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial t}\right)+p\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial t}\right)-\frac{\partial}{\partial\zeta}\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\dot{\zeta}\right)+p\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\dot{\zeta}\right)
\end{aligned}
\tag{5.20}
$$

and the second term in RHS can be manipulated as

$$
p\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial t}\right)=-p\frac{\partial}{\partial\zeta}\left(g\frac{r^2}{a^2}\frac{\partial r}{\partial t}\right)=-p\frac{\partial}{\partial t}\left(g\frac{r^2}{a^2}\frac{\partial r}{\partial\zeta}\right)=p\frac{\partial}{\partial t}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\right)
\tag{5.21}
$$

Then we deal with the three terms with p; the first is Eq. (5.21) which is the second term in Eq. (5.20), and next, the last term in Eq. (5.20), and last, the second term in Eq. (5.19b), so that we obtain

$$
\begin{aligned}
&p\frac{\partial}{\partial t}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\right)+p\nabla_H\bullet\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)+p\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\dot{\zeta}\right)\\
&=p\frac{\kappa h}{p}\left[\frac{\partial}{\partial t}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\right)+\nabla_H\bullet\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}V_H\right)+\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\dot{\zeta}\right)\right]+p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\left[\frac{d}{dt}\left(\frac{\kappa h}{p}\right)\right]\\
&=p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{1}{p^2}\left(p\frac{d\kappa h}{dt}-\kappa h\frac{dp}{dt}\right)=-\frac{1}{\gamma}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{dp}{dt}
\end{aligned}
\tag{5.22}
$$

In summary, Eq. (5.17) can be equal to Eq. (5.19b) and equal to the following after applying Eqs. (5.20), (5.21) and (5.22) as

$$
=-\nabla_H\bullet\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)-\frac{\partial}{\partial\zeta}\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial t}\right)-\frac{\partial}{\partial\zeta}\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\dot{\zeta}\right)-\frac{1}{\gamma}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{dp}{dt}
\tag{5.19c}
$$

Finally, we have the energy conversion term. When we select from Eq. (5.19a) and Eq. (5.19c) with *w* term recovered, we have

$$\frac{1}{\gamma}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{dp}{dt}=-p\nabla_H\bullet\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)+\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\left(w-V_H\bullet\nabla_H r\right)$$
$$-\frac{\partial}{\partial\zeta}\left(p\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\left(w-V_H\bullet\nabla_H r\right)\right) \tag{5.23a}$$

with further manipulation, we have

$$\frac{1}{\gamma}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{dp}{dt}=-p\nabla_H\bullet\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right)-p\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\left(w-V_H\bullet\nabla_H r\right)\right) \tag{5.23b}$$

after some manipulations, then we do a discretization as

$$\left(\frac{dp}{dt}\right)_k=-\gamma_k p_k\left(\nabla_H\bullet\left(V_H\right)+\frac{1}{\Delta r^3}V_H\bullet\nabla_H\Delta r^3-\frac{1}{\Delta r^3}\Delta\left(V_H\bullet\nabla_H r^3\right)+\frac{a^2}{r^2}\frac{\Delta\tilde{w}}{\Delta r}\right)_k \tag{5.24}$$

where

$$\tilde{w}=\frac{r^2}{a^2}w \tag{5.25}$$

is a height-weighted vertical velocity. Finally, when we recover the local symbolic gradients back to their usual form, we have

$$\left(\frac{dp}{dt}\right)_k=-\gamma_k p_k\left\{m^2\left(\frac{\partial}{\partial\lambda}\left(\frac{u^*}{r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{v^*}{r}\right)\right)_k\right.$$
$$-m^2\frac{\left(\frac{u^*_{k-1}}{r_{k-1}}-\frac{u^*_k}{r_k}\right)\frac{\partial\hat{r}^3_k}{\partial\lambda}+\left(\frac{u^*_k}{r_k}-\frac{u^*_{k+1}}{r_{k+1}}\right)\frac{\partial\hat{r}^3_{k+1}}{\partial\lambda}+\left(\frac{v^*_{k-1}}{r_{k-1}}-\frac{v^*_k}{r_k}\right)\frac{\partial\hat{r}^3_k}{\partial\varphi}+\left(\frac{v^*_k}{r_k}-\frac{v^*_{k+1}}{r_{k+1}}\right)\frac{\partial\hat{r}^3_{k+1}}{\partial\varphi}}{2\Delta r^3}$$
$$\left.+\left(\frac{a^2}{r^2}\frac{\Delta\tilde{w}}{\Delta r}\right)_k\right\} \tag{5.26}$$

using the above discretization, which is obtained from kinetic equation, with the thermodynamic equation will ensure total energy conservation.

5.3 Considering potential enthalpy conservation

Since we are using enthalpy as a prognostic equation, the potential enthalpy conservation, see Eqs. (2.10) and (2.11) in section 2, has to be considered which can be written in logarithmic form as

$$\frac{1}{\Theta}\frac{d\Theta}{dt}=\frac{d}{dt}\left(\ln\Theta\right)=0 \tag{5.27}$$

Combining with continuity equation we have

$$\frac{\partial}{\partial t}\left(\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\ln\Theta\right)+m^2\frac{\partial}{\partial\lambda}\left(\frac{u^*}{r}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\ln\Theta\right)+m^2\frac{\partial}{\partial\varphi}\left(\frac{v^*}{r}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\ln\Theta\right)+\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\ln\Theta\right)=0 \tag{5.28}$$

Since horizontal is spectral computation, we do vertical discretization by equaling two vertical terms before combination and after combination as

$$\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\ln\Theta\right)=\left(\dot{\zeta}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\right)\frac{\partial}{\partial\zeta}\left(\ln\Theta\right)+\left(\ln\Theta\right)\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\frac{\partial\tilde{\bar{p}}}{\partial\zeta}\right) \tag{5.29}$$

So vertical advection can be discretized, see Juang (2005, 2011) for details, as following

$$\left(\dot{\zeta}\frac{\partial}{\partial\zeta}\langle\ln\Theta\rangle\right)_k=\frac{\left(\ln\Theta_{k-1}-\ln\Theta_k\right)}{2\left(\hat{\tilde{\bar{p}}}_k-\hat{\tilde{\bar{p}}}_{k+1}\right)}\left(\dot{\zeta}\frac{\partial\hat{\tilde{\bar{p}}}}{\partial\zeta}\right)_k+\frac{\left(\ln\Theta_k-\ln\Theta_{k+1}\right)}{2\left(\hat{\tilde{\bar{p}}}_k-\hat{\tilde{\bar{p}}}_{k+1}\right)}\left(\dot{\zeta}\frac{\partial\hat{\tilde{\bar{p}}}}{\partial\zeta}\right)_{k+1} \tag{5.30}$$

And it is the same for any variable in terms of no forcing in mass weighted advection. Since $h=\Theta\pi$, so $\ln h=\ln\Theta+\ln\pi$ is a linear computation. Next let's start from Eqs. (2.9), (2.10), and (2.11), with conservation in terms of no external forcing as Q=0 and $\frac{d\kappa}{dt}=0$, so the thermodynamic equation can be

$$\frac{dh}{dt}=\frac{\partial}{\partial t}\left(\Theta\pi\right)+m^2\vec{V}\bullet\nabla\left(\Theta\pi\right)+\dot{\zeta}\frac{\partial h}{\partial\zeta}=\frac{\kappa h}{p}\frac{dh}{dt} \tag{5.31}$$

Then we expand the equation into potential enthalpy, pressure and kappa, then apply total derivatives of potential enthalpy and kappa are zero, we have

$$-\pi\dot{\zeta}\frac{\partial\Theta}{\partial\zeta}-h\ln\frac{p}{p_0}\dot{\zeta}\frac{\partial\kappa}{\partial\zeta}+h\kappa\left[\frac{\partial}{\partial t}\left(\ln\frac{p}{p_0}\right)+m^2V\bullet\nabla\left(\ln\frac{p}{p_0}\right)\right]+\dot{\zeta}\frac{\partial h}{\partial\zeta}=\frac{\kappa h}{p}\frac{dp}{dt} \tag{5.32}$$

Move all terms at LHS except vertical advection of h to RHS, expand total derivative of p into logorithms of p, then we obtain the equation for vertical advection of h as

$$\dot{\zeta}\frac{\partial h}{\partial\zeta}=h\dot{\zeta}\frac{\partial}{\partial\zeta}\left(\ln\Theta\right)+h\ln\frac{p}{p_0}\dot{\zeta}\frac{\partial\kappa}{\partial\zeta}+h\kappa\dot{\zeta}\frac{\partial}{\partial\zeta}\left(\ln\frac{p}{p_0}\right) \tag{5.32}$$

Then we discretize the vertical advections in LHS by the same form as Eq. (5.30), so the vertical advection of h can be

$$\left(\dot{\zeta}\frac{\partial h}{\partial\zeta}\right)_k=\frac{h_k}{2\Delta\tilde{\bar{p}}_k}\left[\left(\dot{\zeta}\frac{\partial\hat{\tilde{\bar{p}}}}{\partial\zeta}\right)_k\left(\ln\frac{h_{k-1}}{h_k}-\left(\kappa_{k-1}-\kappa_k\right)\ln\frac{p_{k-1}}{p_k}\right)+\left(\dot{\zeta}\frac{\partial\hat{\tilde{\bar{p}}}}{\partial\zeta}\right)_{k+1}\left(\ln\frac{h_k}{h_{k+1}}+\left(\kappa_k-\kappa_{k+1}\right)\ln\frac{p_k}{p_{k+1}}\right)\right] \tag{5.33}$$

for a Eulerian system to conserve potential enthalpy.

If it is in a dimensional-split semi-Lagrangian system, Eq. (5.33) can be rewritten with further discretization as

$$\dot{\zeta}\frac{\partial h}{\partial\zeta}=h\left\{\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\ln\Theta\right)+\ln\frac{p}{p_0}\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\kappa\right)+\kappa\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\ln\frac{p}{p_0}\right)-\left(\ln\Theta+2\kappa\ln\frac{p}{p_0}\right)\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\delta\right)\right\} \tag{5.34}$$

where $\delta=1$, it indicates that we can apply the flux form semi-Lagrangian advections with a correction of the unit flux form semi-Lagrangian advection. Thus, to have potential enthalpy conservation, we need do vertical flux form semi-Lagrangian advections of logarithms of potential enthalpy, kappa, logarithms of pressure, and a unit value. For three-dimensional advection, to have conservation in advection may requires three dimensional flux form semi-Lagrangian advection.

5.4 Coordinate vertical velocity

From the height-weighted coordinate vertical velocity in Eq. (5.25), we can have the vertical momentum equation use a height-weighted coordinate vertical velocity. Let's start with the vertical momentum equation (2.14c) and apply Eq. (4.5), then we have

$$\begin{aligned}\frac{dw}{dt} &= m^2 \frac{s^{*2}}{r} + m^2 f_c^* u^* - \frac{\kappa h}{p}\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r} - g \\ &\cong m^2 \frac{s^{*2}}{r} + m^2 f_c^* u^* + \bar{g}\frac{r^2 \Delta p}{a^2 \Delta \tilde{\bar{p}}} - g\end{aligned} \tag{5.35}$$

Then the total derivative of height-weighted coordinate vertical velocity will be

$$\begin{aligned}\frac{d\tilde{w}}{dt} &= \frac{r^2}{a^2}\frac{dw}{dt} + 2w\frac{r}{a^2}\frac{dr}{dt} \\ &= 2a^2 \frac{\tilde{w}^2}{r^3} + m^2 \frac{r}{a^2} s^{*2} + m^2 \frac{r^2}{a^2} f_c^* u^* + \bar{g}\left(\frac{r^4 \Delta p}{a^4 \Delta \tilde{\bar{p}}} - 1\right)\end{aligned} \tag{5.36}$$

Furthermore, do a total derivative of Eq. (5.10) so we have

$$\begin{aligned} r_k^2 w_k &= \frac{1}{2}\hat{r}_{k+1}^2 \hat{w}_{k+1} + \frac{1}{2}\hat{r}_k^2 \hat{w}_k \\ \tilde{w}_k &= \frac{1}{2}\hat{\tilde{w}}_{k+1} + \frac{1}{2}\hat{\tilde{w}}_k \end{aligned} \tag{5.37}$$

with the bottom boundary condition as

$$\begin{aligned}\hat{w}_1 &= m^2 \frac{u_1^*}{\hat{r}_1}\frac{\partial \hat{r}_1}{\partial \lambda} + m^2 \frac{v_1^*}{\hat{r}_1}\frac{\partial \hat{r}_1}{\partial \varphi} \\ \hat{\tilde{w}}_1 &= m^2 \frac{\hat{r}_1}{a}\left(u_1^* \frac{\partial \hat{r}_1}{a\partial \lambda} + v_1^* \frac{\partial \hat{r}_1}{a\partial \varphi}\right)\end{aligned} \tag{5.38}$$

Though our prognostic equation for vertical velocity is at model layers, we need it to be at model levels for computation of the thermodynamic equation. We get vertical motion at model levels by Eqs. (5.38) and (5.37) for vertical motion at model layers.

6. Linearization for semi-implicit scheme

From the previous section, we can have all discretized prognostic equations linearized, so that we can use the semi-implicit form in wave space. The linearization can be done along a base state, which is defined and obtained in Appendix B.

6.1 generalized linear equations

First, we look at the horizontal momentum equations (2.14), which have linear terms from the pressure gradient force in Eq. (5.14) as

$$
\begin{aligned}
\left(\frac{du_k^*}{dt}\right)_L = &-\frac{\kappa_{0k}h_{0k}}{\varepsilon_{0k}p_{0k}}\frac{\partial p_k}{a\partial\lambda}+\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \kappa_{0k}h_{0k}\frac{\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}^2}\frac{\partial p_k}{a\partial\lambda}+\sum_{i=1}^{k-1}\kappa_{0i}h_{0i}\frac{\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}^2}\frac{\partial p_i}{a\partial\lambda}\right\rangle \\
&-\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_{0k}\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}}\frac{\partial h_k}{a\partial\lambda}+\sum_{i=1}^{k-1}\frac{\kappa_{0i}\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}}\frac{\partial h_i}{a\partial\lambda}\right\rangle \qquad (6.1a)\\
&-\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_k h_k}{2p_{0k}}\frac{\partial\Delta\tilde{\tilde{p}}_k}{a\partial\lambda}+\sum_{i=1}^{k-1}\frac{\kappa_{0i}h_{0i}}{p_{0i}}\frac{\partial\Delta\tilde{\tilde{p}}_i}{a\partial\lambda}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{dv_k^*}{dt}\right)_L = &-\frac{\kappa_{0k}h_{0k}}{\varepsilon_{0k}p_{0k}}\frac{\partial p_k}{a\partial\varphi}+\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \kappa_{0k}h_{0k}\frac{\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}^2}\frac{\partial p_k}{a\partial\varphi}+\sum_{i=1}^{k-1}\kappa_{0i}h_{0i}\frac{\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}^2}\frac{\partial p_i}{a\partial\varphi}\right\rangle \\
&-\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_{0k}\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}}\frac{\partial h_k}{a\partial\varphi}+\sum_{i=1}^{k-1}\frac{\kappa_{0i}\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}}\frac{\partial h_i}{a\partial\varphi}\right\rangle \qquad (6.1b)\\
&-\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_k h_k}{2p_{0k}}\frac{\partial\Delta\tilde{\tilde{p}}_k}{a\partial\varphi}+\sum_{i=1}^{k-1}\frac{\kappa_{0i}h_{0i}}{p_{0i}}\frac{\partial\Delta\tilde{\tilde{p}}_i}{a\partial\varphi}\right\rangle
\end{aligned}
$$

these can be combined into divergence and vorticity as

$$
\begin{aligned}
\left(\frac{dD_k^*}{dt}\right)_L = &-\frac{\kappa_{0k}h_{0k}}{\varepsilon_{0k}p_{0k}}\nabla^2 p_k+\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \kappa_{0k}h_{0k}\frac{\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}^2}\nabla^2 p_k+\sum_{i=1}^{k-1}\kappa_{0i}h_{0i}\frac{\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}^2}\nabla^2 p_i\right\rangle \\
&-\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_{0k}\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}}\nabla^2 h_k+\sum_{i=1}^{k-1}\frac{\kappa_{0i}\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}}\nabla^2 h_i\right\rangle \qquad (6.2)\\
&-\frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_k h_k}{2p_{0k}}\nabla^2\Delta\tilde{\tilde{p}}_k+\sum_{i=1}^{k-1}\frac{\kappa_{0i}h_{0i}}{p_{0i}}\nabla^2\Delta\tilde{\tilde{p}}_i\right\rangle \\
\left(\frac{d\varsigma_k^*}{dt}\right)_L = &\,0
\end{aligned}
$$

where

$$
\nabla^2 = \frac{1}{a^2}\left(\frac{\partial^2}{\partial\lambda^2}+\frac{\partial^2}{\partial\varphi^2}\right) \qquad (6.3a)
$$

$$
\begin{aligned}
D_k^* &= \left(\frac{\partial u^*}{a\partial\lambda}+\frac{\partial v^*}{a\partial\varphi}\right)_k \\
\varsigma_k^* &= \left(\frac{\partial v^*}{a\partial\lambda}-\frac{\partial u^*}{a\partial\varphi}\right)_k
\end{aligned} \qquad (6.3b)
$$

which are the same definition as in hydrostatic system. So we can use existing spectral transform routines with divergence and vorticity computation as in the shallow atmosphere for computation of Eq. (6.3) here.

Before we can further linearize the vertical momentum and thermodynamics equations, we found they have vertical differences of pressure and vertical velocity in the linear forcing. The vertical differences of pressure and vertical velocity at a model layer are the differences between the two neighboring levels. Since all linearized terms are computed at the model layers, we have to make the differences of pressure and vertical velocity from model level be represented by themselves at the model layer. To do so, let's start with the relationship between levels and layers for pressure first. From $\Delta p_k = \hat{p}_k - \hat{p}_{k+1}$ and $p_k = \frac{1}{2}\left(\hat{p}_k + \hat{p}_{k+1}\right)$, we get that any given pressure at a model layer is given by

$$p_k = \sum_{i=K}^{k+1} \Delta p_i + \frac{1}{2}\Delta p_k = \mathrm{P}_{ki}\Delta p_i \tag{6.4}$$

the same for vertical velocity with

$$\tilde{w}_k = \hat{\tilde{w}}_1 + \sum_{i=1}^{k-1} \Delta\tilde{w}_i + \frac{1}{2}\Delta\tilde{w}_k = \hat{\tilde{w}}_1 + W_{ki}\Delta\tilde{w}_i \tag{6.5}$$

So that we can have differences with the inverse of matrices P and W as

$$\Delta p_k = \mathrm{P}_{ki}^{-1} p_i \tag{6.6}$$

$$\Delta\tilde{w}_k = W_{ki}^{-1}\left(\tilde{w}_i - \hat{\tilde{w}}_1\right) \tag{6.7}$$

In this case, all vertical differencing at two neighboring model levels can be represented by a matrix summation of its value at the model layers. Thus, the vertical momentum equation, Eq. (5.36), can be linearized as

$$\begin{aligned}\left(\frac{d\tilde{w}_k}{dt}\right)_L &= \frac{\overline{g}\varepsilon_{0k}^4}{\Delta\tilde{\overline{p}}_{0k}}\Delta p_k - \frac{\overline{g}}{\Delta\tilde{\overline{p}}_{0k}}\Delta\tilde{\overline{p}}_k \\ &= \frac{\overline{g}\varepsilon_{0k}^4}{\Delta\tilde{\overline{p}}_{0k}}\mathrm{P}_{ki}^{-1}p_i - \frac{\overline{g}}{\Delta\tilde{\overline{p}}_{0k}}\Delta\tilde{\overline{p}}_k\end{aligned} \tag{6.8}$$

And the pressure and the thermodynamic equations, see Eq. (5.26), can be linearized as

$$\begin{aligned}\left(\frac{dp_k}{dt}\right)_L &= -\frac{\gamma_{0k}p_{0k}}{\varepsilon_{0k}}D_k^* + \frac{\gamma_{0k}\overline{g}p_{0k}^2}{\kappa_{0k}h_{0k}\Delta\tilde{\overline{p}}_{0k}}\left(\Delta\tilde{w}_k\right)_L \\ &= -\frac{\gamma_{0k}p_{0k}}{\varepsilon_{0k}}D_k + \frac{\gamma_{0k}\overline{g}p_{0k}^2}{\kappa_{0k}h_{0k}\Delta\tilde{\overline{p}}_{0k}}W_{ki}^{-1}\tilde{w}_i\end{aligned} \tag{6.9}$$

$$\left(\frac{dh_k}{dt}\right)_L = -\frac{\gamma_{0k}\kappa_{0k}h_{0k}}{\varepsilon_{0k}}D_k + \frac{\gamma_{0k}\overline{g}p_{0k}}{\Delta\tilde{\overline{p}}_{0k}}W_{ki}^{-1}\tilde{w}_i \tag{6.10}$$

Then, the remaining equation is the continuity equation. Since we are going to use a semi-Lagrangian scheme, let's have the total derivative as follows for linearization

$$\frac{d_h}{d_h t}\Delta\tilde{\overline{p}}_k + m^2\Delta\tilde{\overline{p}}_k\left(\frac{\partial}{\partial\lambda}\left(\frac{u^*}{r}\right) + \frac{\partial}{\partial\varphi}\left(\frac{v^*}{r}\right)\right)_k + \left(\widehat{\dot{\zeta}\frac{\partial\tilde{\overline{p}}}{\partial\zeta}}\right)_k - \left(\widehat{\dot{\zeta}\frac{\partial\tilde{\overline{p}}}{\partial\zeta}}\right)_{k+1} = 0 \tag{6.11}$$

For a specific coordinate, we may be able to linearize the vertical fluxes, last two terms in the above equation, but for generalization and simplicity, we do divergence only here, and do the rest in the next section as

$$\left(\frac{d_h \Delta\tilde{\bar{p}}_k}{d_h t}\right)_L = -\Delta\tilde{\bar{p}}_{0k}\frac{a}{r_{0k}}\left(\frac{\partial u^*}{a\partial\lambda}+\frac{\partial v^*}{a\partial\varphi}\right)_k = -\frac{\Delta\tilde{\bar{p}}_{0k}}{\varepsilon_{0k}}D_k^* \tag{6.12}$$

Thus, we have linearized equations, Eqs. (6.2), (6.8), (6.9), (6.10) and (6.12), a for semi-implicit time integration scheme.

6.2 Linearization along sigma-pressure coordinates

For a backward comparison similar to the hydrostatic system, and from Eq. (4.5) we know that the coordinate pressure is monotonic, thus, we can use it for the vertical coordinate by predefining it as

$$\hat{\tilde{\bar{p}}}_k = \hat{A}_k + \hat{B}_k \hat{\tilde{\bar{p}}}_s \tag{6.13}$$

Thus, the previous generalized linearized equations will turn into following

$$\begin{aligned}\left(\frac{dD_k^*}{dt}\right)_L =& -\frac{\kappa_{0k}h_{0k}}{\varepsilon_{0k}p_{0k}}\nabla^2 p_k + \frac{\Delta p_{0k}}{\Delta\tilde{\bar{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \kappa_{0k}h_{0k}\frac{\Delta\tilde{\bar{p}}_{0k}}{2p_{0k}^2}\nabla^2 p_k + \sum_{i=1}^{k-1}\kappa_{0i}h_{0i}\frac{\Delta\tilde{\bar{p}}_{0i}}{p_{0i}^2}\nabla^2 p_i\right\rangle \\ & -\frac{\Delta p_{0k}}{\Delta\tilde{\bar{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_{0k}\Delta\tilde{\bar{p}}_{0k}}{2p_{0k}}\nabla^2 h_k + \sum_{i=1}^{k-1}\frac{\kappa_{0i}\Delta\tilde{\bar{p}}_{0i}}{p_{0i}}\nabla^2 h_i\right\rangle \\ & -\frac{\Delta p_{0k}}{\Delta\tilde{\bar{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_k h_k \Delta B_k}{2p_{0k}} + \sum_{i=1}^{k-1}\frac{\kappa_{0i}h_{0i}\Delta B_{0i}}{p_{0i}}\right\rangle\nabla^2\tilde{\bar{p}}_s\end{aligned} \tag{6.14}$$

$$\left(\frac{d\tilde{w}_k}{dt}\right)_L = \frac{\bar{g}\varepsilon_{0k}^4}{\Delta\tilde{\bar{p}}_{0k}}\mathrm{P}_{ki}^{-1}p_i - \frac{\bar{g}}{\Delta\tilde{\bar{p}}_{0k}}\Delta B_k \hat{\tilde{\bar{p}}}_s \tag{6.15}$$

with a total vertical summation of all layers for Eq. (6.11), we have

$$\frac{\partial\tilde{\bar{p}}_s}{\partial t} + m^2\sum_{k=1}^{K}\frac{\Delta B_k}{\varepsilon_k}\left(u_k^*\frac{\partial\tilde{\bar{p}}_s}{a\partial\lambda}+v_k^*\frac{\partial\tilde{\bar{p}}_s}{a\partial\varphi}\right) + m^2\sum_{k=1}^{K}\Delta\tilde{\bar{p}}_k\left(\frac{\partial}{a\partial\lambda}\left(\frac{u^*}{r}\right)+\frac{\partial}{a\partial\varphi}\left(\frac{v^*}{r}\right)\right)_k = 0 \tag{6.16}$$

and the linearized form is

$$\left(\frac{D\tilde{\bar{p}}_s}{Dt}\right)_L = -\sum_{k=1}^{K}\frac{\Delta\tilde{\bar{p}}_{0k}}{\varepsilon_{0k}}D_k^* \tag{6.17}$$

where

$$\frac{D\tilde{\bar{p}}_s}{Dt} = \frac{\partial\tilde{\bar{p}}_s}{\partial t} + m^2\left(U_k^*\frac{\partial\tilde{\bar{p}}_s}{a\partial\lambda}+V_k^*\frac{\partial\tilde{\bar{p}}_s}{a\partial\varphi}\right) \tag{6.18}$$

and

$$\begin{aligned}U_k^* &= \sum_{k=1}^{K}\frac{\Delta B_k u_k^*}{\varepsilon_k} \\ V_k^* &= \sum_{k=1}^{K}\frac{\Delta B_k v_k^*}{\varepsilon_k}\end{aligned} \tag{6.19}$$

is the total vertical contribution of advection to the surface coordinate pressure. Thus, for the linearization along a sigma-pressure coordinate, we need Eqs. (6.14), (6.15), and (6.17), which change into using coordinate surface pressure, and Eqs. (6.9) and (6.10) to be completed.

7. Applying a SETTLS scheme for semi-implicit and semi-Lagrangian time integration

There are several semi-implicit and semi-Lagrangian time integration schemes in the literature. Since we have a SETTLS scheme in the NCEP GSM code, we can derive the solution of the semi-implicit and semi-Lagrangian methods along the lines of a SETTLS scheme, however, the manipulation and final computation sequences in this section can be used for any time scheme.

First, let's introduce the formula of the SETTLS scheme. It can be found in Hortel (1999), and we give details here. Any location along the advection can be expanded by a Taylor series as

$$f(x+\Delta x, t+\delta t) = f(x,t) + \left(\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial t}\delta t \right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial t^2}(\delta t)^2 + \frac{\partial^2 f}{\partial x \partial t}\Delta x \delta t \right) + .. \quad (7.1)$$

where the second term on the RHS can be replaced by $\frac{df}{dt}$ along the velocity of $u = \frac{\Delta x}{\delta t}$, and the third term can be approximated to $\frac{d^2 f}{dt^2}$ at $x + \frac{1}{2}\Delta x$ and $t + \frac{1}{2}\delta t$, thus, we have

$$\begin{aligned}
f_{x+\Delta x}^{t+\delta t} &\approx f_x^t + \delta t \left(\frac{df}{dt} \right)_x^t + \frac{(\delta t)^2}{2!} \left(\frac{d^2 f}{dt^2} \right)_{x+\frac{1}{2}\Delta x}^{t+\frac{1}{2}\delta t} \\
&\approx f_x^t + \delta t \left(\frac{df}{dt} \right)_x^t + \frac{(\delta t)^2}{2!} \left(\frac{d^2 f}{dt^2} \right)_{x+\frac{1}{2}\Delta x}^{t-\frac{1}{2}\delta t} \\
&\approx f_x^t + \delta t \left(\frac{df}{dt} \right)_x^t + \frac{(\delta t)^2}{2!} \frac{\left(\frac{df}{dt} \right)_{x+\Delta x}^t - \left(\frac{df}{dt} \right)_x^{t-\delta t}}{\delta t} \\
&\approx f_x^t + \frac{\delta t}{2} \left(2\left(\frac{df}{dt} \right)_x^t - \left(\frac{df}{dt} \right)_x^{t-\delta t} + \left(\frac{df}{dt} \right)_{x+\Delta x}^t \right)
\end{aligned} \quad (7.2)$$

Based on this approximation, we can summarize the advection with the following

$$\begin{aligned}
x_A^{n+1} &\approx x_D^n + \delta t V_M^{n+\frac{1}{2}} \\
&\approx x_D^n + \delta t \frac{V_A^n + 2V_D^n - V_D^{n-1}}{2}
\end{aligned} \quad (7.3)$$

which determines the location of the departure and arrival locations with velocities at the departure and arrival locations for current and past times. Any total derivative of all prognostic variables and their associated Lagrangian forcing (source terms) can be the same as x and V, respectively. Thus, any given prognostic equation can be illustrated as follows for this forcing method. For any given prognostic equation we have

$$\frac{dA}{dt} = F = L + N \tag{7.4}$$

where A is any prognostic variable, F is its Lagrangian forcing which can be separated into linear terms L and nonlinear terms N. So the semi-Lagrangian scheme can be written and expanded by the above scheme as

$$\begin{aligned} A_A^{n+1} &= A_D^n + \delta t\left(F - L\right)_M^{n+\frac{1}{2}} + \delta t L \\ &= A_D^n + \delta t\left(\alpha L_A^{n+1} + (1-\alpha)L_D^n - L_M^{n+\frac{1}{2}}\right) + \delta t F_M^{n+\frac{1}{2}} \\ &= A_D^n + \delta t\left(\alpha L_A^{n+1} + (1-\alpha)L_D^n - \alpha L_A^n - (1-\alpha)L_D^{n+1}\right) + \delta t\frac{F_A^n + F_D^{n+1}}{2} \\ &= A_D^n + \delta t\left(\alpha\left\langle L_A^{n+1} - L_A^n\right\rangle - (1-\alpha)\left\langle L_D^n - L_D^{n-1}\right\rangle\right) + \delta t\frac{F_A^n + 2F_D^n - F_D^{n-1}}{2} \end{aligned} \tag{7.5}$$

where an un-centered option is used for the linear terms by α. With rearrangement to group departure and arrival terms together and put the unknown term onto the LHS of Eq. (7.5), we have

$$\delta A_A^{n+1} - \alpha\delta t\delta L_A^{n+1} = A_D^n - (1-\alpha)\delta t\left(L_D^n - L_D^{n-1}\right) + \delta t\frac{2F_D^n - F_D^{n-1}}{2} + \delta t\frac{F_A^n}{2} - A_A^n = S_A^n \tag{7.6}$$

where

$$\delta A_A^{n+1} = A_A^{n+1} - A_A^n \tag{7.7a}$$

$$\delta L_A^{n+1} = L_A^{n+1} - L_A^n \tag{7.7b}$$

Then we can apply Eq. (7.6) to all prognostic equations with linear equations from Eqs. (6.14), (6.15), (6.10), (6.9) and (6.17), respectively, as follows:

$$\delta D_k^* - \frac{n(n+1)}{a^2}\alpha\delta t\left(\mathrm{A}_{ki}\delta p_i + \mathrm{B}_{ki}\delta h_i + e_{0k}\delta\tilde{\tilde{p}}_s\right) = S_{D*} \tag{7.8a}$$

$$\delta\tilde{w}_k + \alpha\delta t\left(b_{0k}\delta\tilde{\tilde{p}}_s - \Gamma_{ki}\delta p_i\right) = S_{\tilde{w}_k} \tag{7.8b}$$

$$\delta h_k + \alpha\delta t\left(f_{0k}\delta D_k^* - \mathrm{Z}_{ki}\delta\tilde{w}_i\right) = S_{h_k} \tag{7.8c}$$

$$\delta p_k + \alpha\delta t\left(d_{0k}\delta D_k^* - \mathrm{M}_{ki}\delta\tilde{w}_i\right) = S_{p_k} \tag{7.8d}$$

$$\delta\tilde{\tilde{p}}_s + \alpha\delta t\Pi_{1i}\delta D_i^* = S_{\tilde{\tilde{p}}_s} \tag{7.8e}$$

where all S terms are computed following Eq. (7.6) by semi-Lagrangian advection through the SETTLS scheme. All matrices in Eq. (7.8) for a given layer are

$$\mathrm{A}_{ki} = \frac{\kappa_{0k}h_{0k}}{\varepsilon_{0k}p_{0k}} - \frac{\Delta p_{0k}}{\Delta\tilde{\tilde{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle\kappa_{0k}h_{0k}\frac{\Delta\tilde{\tilde{p}}_{0k}}{2p_{0k}^2} + \sum_{i=1}^{k-1}\kappa_{0i}h_{0i}\frac{\Delta\tilde{\tilde{p}}_{0i}}{p_{0i}^2}\right\rangle \tag{7.9a}$$

$$\mathrm{B}_{ki} = \frac{\Delta p_{0k}}{\Delta\tilde{\bar{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_{0k}\Delta\tilde{\bar{p}}_{0k}}{2p_{0k}} + \sum_{i=1}^{k-1}\frac{\kappa_{0i}\Delta\tilde{\bar{p}}_{0i}}{p_{0i}} \right\rangle \tag{7.9b}$$

$$\Gamma_{ki} = \frac{\bar{g}\varepsilon_{0k}^{4}}{\Delta\tilde{\bar{p}}_{0k}}\mathrm{P}_{ki}^{-1} \tag{7.9c}$$

$$Z_{ki} = \frac{\gamma_{0k}\bar{g}p_{0k}}{\Delta\tilde{\bar{p}}_{0k}}W_{ki}^{-1} \tag{7.9d}$$

$$\mathrm{M}_{ki} = \frac{\gamma_{0k}\bar{g}p_{0k}^{2}}{\kappa_{0k}h_{0k}\Delta\tilde{\bar{p}}_{0k}}\mathrm{P}_{ki}^{-1} \tag{7.9e}$$

and the vector in Eq. (7.8e) is

$$\Pi_{1i} = \sum_{i=1}^{K}\frac{\Delta\tilde{\bar{p}}_{0i}}{\varepsilon_{0i}} \tag{7.9f}$$

and the constants for a given layer k are

$$e_{0k} = \frac{\Delta p_{0k}}{\Delta\tilde{\bar{p}}_{0k}}\frac{1}{\varepsilon_{0k}}\left\langle \frac{\kappa_{k}h_{k}\Delta B_{k}}{2p_{0k}} + \sum_{i=1}^{k-1}\frac{\kappa_{0i}h_{0i}\Delta B_{0i}}{p_{0i}} \right\rangle \tag{7.9g}$$

$$b_{0k} = \frac{\bar{g}}{\Delta\tilde{\bar{p}}_{0k}}\Delta B_{k} \tag{7.9f}$$

$$f_{0k} = \frac{\gamma_{0k}\kappa_{0k}h_{0k}}{\varepsilon_{0k}} \tag{7.9h}$$

$$d_{0k} = \frac{\gamma_{0k}p_{0k}}{\varepsilon_{0k}} \tag{7.9i}$$

The details of the matrices, vectors, and constants are given in Appendix C with an example for 6 layers.

Using Eq. (7.8), we can start to solve for the n+1 values by eliminating variables in spectral space. First, we put Eqs. (7.8d) and (7.8e) into Eq. (7.8b) to solve the w equation as

$$\delta\tilde{w}_{k} = \left[I - (\alpha\delta t)^{2}\Gamma_{ki}\mathrm{M}_{ij}\right]^{-1}\left[(\alpha\delta t)^{2}\left(b_{0k}\Pi_{1i} + \Gamma_{ki}d_{0i}\right)\delta D_{i}^{*} + S_{\tilde{w}_{k}} - \alpha\delta t\left(b_{0k}S_{\tilde{\bar{p}}_{s}} - \Gamma_{ki}S_{p_{i}}\right)\right] \tag{7.10}$$

then by putting Eq. (7.10) into Eqs. (7.8c) and (7.8d), we have

$$\begin{aligned}\delta h_{k} &= S_{h_{k}} - \alpha\delta t f_{0k}\delta D_{k}^{*} + \alpha\delta t \mathrm{Z}_{ki} \\ &\quad \left[I - (\alpha\delta t)^{2}\Gamma_{ki}\mathrm{M}_{ij}\right]^{-1}\left[(\alpha\delta t)^{2}\left(b_{0k}\Pi_{1i} + \Gamma_{ki}d_{0i}\right)\delta D_{i}^{*} + S_{\tilde{w}_{k}} - \alpha\delta t\left(b_{0k}S_{\tilde{\bar{p}}_{s}} - \Gamma_{ki}S_{p_{i}}\right)\right]\end{aligned} \tag{7.11}$$

$$\begin{aligned}\delta p_{k} &= S_{p_{k}} - \alpha\delta t d_{0k}\delta D_{k}^{*} + \alpha\delta t \mathrm{M}_{ki} \\ &\quad \left[I - (\alpha\delta t)^{2}\Gamma_{ki}\mathrm{M}_{ij}\right]^{-1}\left[(\alpha\delta t)^{2}\left(b_{0k}\Pi_{1i} + \Gamma_{ki}d_{0i}\right)\delta D_{i}^{*} + S_{\tilde{w}_{k}} - \alpha\delta t\left(b_{0k}S_{\tilde{\bar{p}}_{s}} - \Gamma_{ki}S_{p_{i}}\right)\right]\end{aligned} \tag{7.12}$$

Finally, we put Eqs. (7.11), (7.12) and (7.8e) into Eq. (7.8a), and we get

$$\delta D_k^* = \left(\mathrm{I}+\Omega\right)^{-1}\Big\{S_{D_k^*} + \frac{n(n+1)}{a^2}\alpha\delta t\Big\langle\left(\mathrm{A}_{ki}S_{p_i} + \mathrm{B}_{ki}S_{h_i} + e_{0k}S_{\tilde{\bar{p}}_s}\right)$$
$$+\alpha\delta t\left(\mathrm{A}_{ki}\mathrm{M}_{ij} + \mathrm{B}_{ki}\mathrm{Z}_{ij}\right)\left[I-(\alpha\delta t)^2\Gamma_{jl}\mathrm{M}_{lk}\right]_{kj}^{-1}\left(S_{\tilde{w}_j} - \alpha\delta t\left(b_{0j}S_{\tilde{\bar{p}}_s} - \Gamma_{jl}S_{p_l}\right)\right)\Big\rangle\Big\} \tag{7.13}$$

where

$$\Omega = \frac{n(n+1)}{a^2}(\alpha\delta t)^2\Big\{\left(\mathrm{A}_{ki}d_{0i} + \mathrm{B}_{ki}f_{0i} + e_{0k}\Pi_{1i}\right)$$
$$-(\alpha\delta t)^2\left(\mathrm{A}_{ki}\mathrm{M}_{ij} + \mathrm{B}_{ki}\mathrm{Z}_{ij}\right)\left[I-(\alpha\delta t)^2\Gamma_{jl}\mathrm{M}_{lk}\right]^{-1}\left(b_{0j}\Pi_{1l} + \Gamma_{jl}d_{0l}\right)\Big\} \tag{7.14}$$

After we obtain the divergence change in Eq. (7.13), we can get the coordinate vertical velocity change from Eq. (7.10) with a divergence change. Then by using the divergence change and coordinate vertical velocity change, we can solve for h and pressure changes with Eqs. (7.8c) and (7.8d), and the coordinate pressure change can be solved by Eq. (7.8e) by a divergence change. With all variable changes solved, the n+1 prognostic variables are obtained.

8. Conclusion and discussion

A vertical discretization of a deep atmospheric nonhydrostatic system has been provided. However, there are several new methods introduced in this note, which are not published in any literature. We don't know whether it is a proper method for deep atmospheric nonhydrostatic dynamics. For example, the mass coordinates defined here are somewhat different from Wood and Staniforth (2003), which were given based on the concept of Laprise's mass conservation. What we introduced here is the concept of mean pressure at any given surface from the air weight on top of the surface, and normalizes it on the mean earth radius with mean gravitational force. Though the forms are slightly different, the mass coordinate concept is the same.

Based on the coordinate pressure, we have found several other variables can be used as height-weighted to simplify the discretization, such as height-weighted coordinate vertical velocity, and height-weighted coordinate density. And the simplicity of the linear average of the values at neighboring model levels can be represented as the value at the model layer, such as with pressure, cubic height, and height-weighted coordinate vertical velocity. All these discretizations are not found in the literature, thus, it will be important to code the NCEP GSM following these discretizations to ensure these discretization equations are done properly.

From the angular momentum and total energy conservation properties, we can have following conclusions; (1) *a deep-atmospheric system "has to be" a nonhydrostatic system with vertical component of Coriolis force*, because *w* is shown in angular momentum conservation property which requires *w* equation and vertical component of Coriolis force; (2) *a hydrostatic system "has to be" shallow atmosphere*, because in hydrostatic system, *w* equation is not used, so vertical component of Coriolis force should not be existed in the horizontal momentum equation, thus *r* has to be constant for angular momrntum conservation equation without *w*; (3) *a nonhydrostaic system "can be" a shallow atmosphere without vertical component of Coriolis force,* because angular momentum has no *w* due to r=a; and (4) *a nonhydrostatic system "can be" a deep atmosphere system with vertical component of Coriolis force*.

Furthermore, in this note, we have not included all possible numerical techniques, such as horizontal diffusion to control short wave noise, divergence damping to control excessive strong wind for Eulerian system and strong wind deformation for semi-Lagrangian system, perturbation on surface pressure advection with related to terrain to avoid orographic resonance, and iteration to solve geopotential height due to the possible unstable from nonhydrostatic mass coordinates used in nonhydrostatic system. We may need some of these techniques after we code the system and will be written in the future note. Nonetheless, this note is sufficient to serve for starting a deep-atmospheric nonhydrostatic dynamics modeling on the sigma-pressure generalized hybrid coordinates.

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Appendix A
Detailed derivation for model-used continuity equation from height coordinate

Converting the density equation in different coordinates for the deep-atmospheric nonhydrostatic dynamics is somewhat tricky. Thus, this appendix will provide a detailed derivation for the density equation from z coordinates to generalized coordinates and to the spherical virtual wind generalized coordinate. The density equation in height coordinates as Eq. (2.6) can be written in direct conversion form as

$$\frac{\partial \rho}{\partial t}-\frac{\frac{\partial \rho}{\partial \zeta}}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial t}+\frac{1}{r\cos\phi}\left(\frac{\partial}{\partial \lambda}(\rho u)-\frac{\frac{\partial}{\partial \zeta}(\rho u)}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial \lambda}+\frac{\partial}{\partial \phi}(\rho v\cos\phi)-\frac{\frac{\partial}{\partial \zeta}(\rho v\cos\phi)}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial \phi}\right)+\frac{1}{r^2}\frac{\frac{\partial}{\partial \zeta}\left(\rho r^2 w\right)}{\frac{\partial r}{\partial \zeta}}=F_\rho \quad \text{(A.1)}$$

which is not the final form because there are two local time derivatives that should be taken care of. To do so, we put w in generalized coordinate form as follows

$$w=\frac{\partial r}{\partial t}+\frac{u}{r\cos\phi}\frac{\partial r}{\partial \lambda}+\frac{v}{r}\frac{\partial r}{\partial \phi}+\dot{\zeta}\frac{\partial r}{\partial \zeta} \quad \text{(A.2)}$$

and into the last term of Eq. (A.1), then we do each dimension separately in following equations, first for the temporal term as

$$\begin{aligned}\frac{1}{r^2}\frac{\frac{\partial}{\partial \zeta}\left\langle \rho r^2\frac{\partial r}{\partial t}\right\rangle}{\frac{\partial r}{\partial \zeta}}&=\frac{\frac{\partial \rho}{\partial \zeta}}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial t}+\frac{\rho}{r^2}\frac{\frac{\partial}{\partial \zeta}\left\langle r^2\frac{\partial r}{\partial t}\right\rangle}{\frac{\partial r}{\partial \zeta}}\\&=\frac{\frac{\partial \rho}{\partial \zeta}}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial t}+\frac{\rho}{r^2}\frac{\frac{\partial}{\partial \zeta}\left\langle \frac{\partial}{\partial t}\left(r^3/3\right)\right\rangle}{\frac{\partial r}{\partial \zeta}}=\frac{\frac{\partial \rho}{\partial \zeta}}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial t}+\frac{\rho}{r^2}\frac{\frac{\partial}{\partial t}\left\langle r^2\frac{\partial r}{\partial \zeta}\right\rangle}{\frac{\partial r}{\partial \zeta}}\end{aligned} \quad \text{(A.3)}$$

the latitudinal terms as

$$\begin{aligned}\frac{1}{r^2}\frac{\frac{\partial}{\partial \zeta}\left\langle \rho r^2\frac{u}{r\cos\phi}\frac{\partial r}{\partial \lambda}\right\rangle}{\frac{\partial r}{\partial \zeta}}&=\frac{1}{r\cos\phi}\frac{\frac{\partial}{\partial \zeta}\langle \rho u\rangle}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial \lambda}+\frac{\rho u}{r^2}\frac{\frac{\partial}{\partial \zeta}\left\langle \frac{r^2}{r\cos\phi}\frac{\partial r}{\partial \lambda}\right\rangle}{\frac{\partial r}{\partial \zeta}}\\&=\frac{1}{r\cos\phi}\frac{\frac{\partial}{\partial \zeta}\langle \rho u\rangle}{\frac{\partial r}{\partial \zeta}}\frac{\partial r}{\partial \lambda}+\frac{\rho u}{r^2}\frac{\frac{\partial}{\partial \lambda}\left\langle \frac{r^2}{r\cos\phi}\frac{\partial r}{\partial \zeta}\right\rangle}{\frac{\partial r}{\partial \zeta}}\end{aligned} \quad \text{(A.4)}$$

the logitudinal terms as

$$\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left\langle\rho r^2\frac{v\cos\phi}{r\cos\phi}\frac{\partial r}{\partial\phi}\right\rangle}{\frac{\partial r}{\partial\zeta}}=\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left\langle\rho v\cos\phi\right\rangle}{\frac{\partial r}{\partial\zeta}}\frac{r^2}{r\cos\phi}\frac{\partial r}{\partial\phi}+\frac{\rho v\cos\phi}{r^2}\frac{\frac{\partial}{\partial\zeta}\left\langle\frac{r^2}{r\cos\phi}\frac{\partial r}{\partial\phi}\right\rangle}{\frac{\partial r}{\partial\zeta}}$$
$$=\frac{1}{r\cos\phi}\frac{\frac{\partial}{\partial\zeta}\left\langle\rho v\cos\phi\right\rangle}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\phi}+\frac{\rho v}{r^2}\frac{\frac{\partial}{\partial\phi}\left\langle\frac{r^2}{r}\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}} \tag{A.5}$$

and the vertical term as

$$\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left\langle\rho r^2\dot{\zeta}\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}}=\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left\langle\rho\dot{\zeta}r^2\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}} \tag{A.6}$$

Then we put Eqs. (A.3)-(A.6) into Eq. (A.1), and we have

$$\frac{\partial\rho}{\partial t}+\frac{\rho}{r^2}\frac{\frac{\partial}{\partial t}\left\langle r^2\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}}+\frac{1}{r\cos\phi}\left(\frac{\partial}{\partial\lambda}\left\langle\rho u\right\rangle+\frac{\partial}{\partial\phi}\left\langle\rho v\cos\phi\right\rangle\right)+$$
$$\frac{\rho u}{r^2}\frac{\frac{\partial}{\partial\lambda}\left\langle\frac{r^2}{r\cos\phi}\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}}+\frac{\rho v}{r^2}\frac{\frac{\partial}{\partial\phi}\left\langle\frac{r^2}{r}\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}}+\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left\langle\rho\dot{\zeta}r^2\frac{\partial r}{\partial\zeta}\right\rangle}{\frac{\partial r}{\partial\zeta}}=F_\rho \tag{A.7}$$

and by multiplying the above with $r^2\frac{\partial r}{\partial\zeta}\cos\phi=\alpha\cos\phi$ we get

$$\alpha\cos\phi\frac{\partial\rho}{\partial t}+\cos\phi\rho\frac{\partial\alpha}{\partial t}+\frac{\alpha}{r}\left(\frac{\partial}{\partial\lambda}\left\langle\rho u\right\rangle+\frac{\partial}{\partial\phi}\left\langle\rho v\cos\phi\right\rangle\right)+$$
$$\rho u\frac{\partial}{\partial\lambda}\left\langle\frac{\alpha}{r}\right\rangle+\rho v\cos\phi\frac{\partial}{\partial\phi}\left\langle\frac{\alpha}{r}\right\rangle+\cos\phi\frac{\partial}{\partial\zeta}\left\langle\rho\dot{\zeta}\alpha\right\rangle=\alpha\cos\phi F_\rho \tag{A.8}$$

let $\beta=\rho r^2\frac{\partial r}{\partial\zeta}\cos\phi=\rho\alpha\cos\phi$ and put it into the above equation, we get the density equation in generalized coordinates as

$$\frac{\partial\beta}{\partial t}+\frac{\partial}{\partial\lambda}\left\langle\beta\dot{\lambda}\right\rangle+\frac{\partial}{\partial\phi}\left\langle\beta\dot{\phi}\right\rangle+\frac{\partial}{\partial\zeta}\left\langle\beta\dot{\zeta}\right\rangle=F_\rho^\beta \tag{A.9}$$

which is Eq. (2.12e). Note that

$$dm=\rho dv=\rho dxdydz=\rho r^2\frac{\partial r}{\partial\zeta}\cos\phi d\lambda d\phi d\zeta=\beta d\lambda d\phi d\zeta \tag{A.10}$$

so that ρ is the density for x, y, and z coordinates and β is the density for λ, ϕ, and ζ coordinates.

It is the same procedure for generalized coordinates with virtual wind. The density equation with virtual wind in the z coordinate can be written as

$$\frac{\partial\rho}{\partial t}+\frac{m^2}{r}\left[\frac{\partial}{\partial\lambda}\left(\rho u^*\right)+\frac{\partial}{\partial\varphi}\left(\rho v^*\right)\right]+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 w\right)=F_\rho \tag{A.11}$$

Using the coordinate conversion, we can have the above equation in direct conversion form as

$$\frac{\partial\rho}{\partial t}-\frac{\frac{\partial\rho}{\partial\zeta}}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial t}+\frac{m^2}{r}\left(\frac{\partial}{\partial\lambda}\left(\rho u^*\right)-\frac{\frac{\partial}{\partial\zeta}\left(\rho u^*\right)}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\lambda}+\frac{\partial}{\partial\varphi}\left(\rho v^*\right)-\frac{\frac{\partial}{\partial\zeta}\left(\rho v^*\right)}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\varphi}\right)+\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left(\rho r^2 w\right)}{\frac{\partial r}{\partial\zeta}}=F_\rho \tag{A.12}$$

again, which is not the final form because there are two local time derivatives that should be taken care of. Again, to do so, we put w into generalized coordinates with virtual wind as follows

$$w=\frac{\partial r}{\partial t}+m^2\frac{u^*}{r}\frac{\partial r}{\partial\lambda}+m^2\frac{v^*}{r}\frac{\partial r}{\partial\varphi}+\dot{\zeta}\frac{\partial r}{\partial\zeta} \tag{A.13}$$

into the last term of Eq. (A.12), then we do each dimension separately in the following equations, first for the temporal term as it is the same as Eq. (A.3) and the third dimension as is the same as in Eq. (A.6); the remainder is the horizontal dimension in Eqs. (A.14) and (A.15).

The latitudinal term as

$$\begin{aligned}\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left(\rho r^2\frac{m^2u^*}{r}\frac{\partial r}{\partial\lambda}\right)}{\frac{\partial r}{\partial\zeta}}&=\frac{m^2}{r}\frac{\frac{\partial}{\partial\zeta}\left(\rho u^*\right)}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\lambda}+\frac{m^2\rho u^*}{r^2}\frac{\frac{\partial}{\partial\zeta}\left(\frac{r^2}{r}\frac{\partial r}{\partial\lambda}\right)}{\frac{\partial r}{\partial\zeta}}\\&=\frac{m^2}{r}\frac{\frac{\partial}{\partial\zeta}\left(\rho u^*\right)}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\lambda}+\frac{m^2\rho u^*}{r^2}\frac{\frac{\partial}{\partial\lambda}\left(\frac{r^2}{r}\frac{\partial r}{\partial\zeta}\right)}{\frac{\partial r}{\partial\zeta}}\end{aligned} \tag{A.14}$$

and longitudinal term next

$$\begin{aligned}\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left(\rho r^2\frac{m^2v^*}{r}\frac{\partial r}{\partial\varphi}\right)}{\frac{\partial r}{\partial\zeta}}&=\frac{m^2}{r}\frac{\frac{\partial}{\partial\zeta}\left(\rho v^*\right)}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\varphi}+\frac{m^2\rho v^*}{r^2}\frac{\frac{\partial}{\partial\zeta}\left(r^2\frac{1}{r}\frac{\partial r}{\partial\varphi}\right)}{\frac{\partial r}{\partial\zeta}}\\&=\frac{m^2}{r}\frac{\frac{\partial}{\partial\zeta}\left(\rho v^*\right)}{\frac{\partial r}{\partial\zeta}}\frac{\partial r}{\partial\varphi}+\frac{m^2\rho v^*}{r^2}\frac{\frac{\partial}{\partial\varphi}\left(\frac{r^2}{r}\frac{\partial r}{\partial\zeta}\right)}{\frac{\partial r}{\partial\zeta}}\end{aligned} \tag{A.15}$$

Then we put Eqs. (A.3), (A.14), (A.15), and (A.6) into Eq. (A.12), and we have

$$\frac{\partial\rho}{\partial t}+\frac{m^2}{r}\left(\frac{\partial}{\partial\lambda}\left(\rho u^*\right)+\frac{\partial}{\partial\varphi}\left(\rho v^*\right)\right)+$$

$$\frac{\rho}{r^2}\frac{\frac{\partial}{\partial t}\left(r^2\frac{\partial r}{\partial\zeta}\right)}{\frac{\partial r}{\partial\zeta}}+\frac{m^2\rho u^*}{r^2}\frac{\frac{\partial}{\partial\lambda}\left(\frac{r^2}{r}\frac{\partial r}{\partial\zeta}\right)}{\frac{\partial r}{\partial\zeta}}+\frac{m^2\rho v^*}{r^2}\frac{\frac{\partial}{\partial\varphi}\left(\frac{r^2}{r}\frac{\partial r}{\partial\zeta}\right)}{\frac{\partial r}{\partial\zeta}}+\frac{1}{r^2}\frac{\frac{\partial}{\partial\zeta}\left(\rho\dot{\zeta}r^2\frac{\partial r}{\partial\zeta}\right)}{\frac{\partial r}{\partial\zeta}}=F_\rho \tag{A.16}$$

when we multiply the above with $\alpha=\frac{r^2}{a^2}\frac{\partial r}{\partial\zeta}$ we get

$$\alpha\frac{\partial\rho}{\partial t}+\rho\frac{\partial\alpha}{\partial t}+m^2\frac{\alpha}{r}\left(\frac{\partial}{\partial\lambda}\left(\rho u^*\right)+\frac{\partial}{\partial\varphi}\left(\rho v^*\right)\right)+$$

$$m^2\rho u^*\frac{\partial}{\partial\lambda}\left(\frac{\alpha}{r}\right)+m^2\rho v^*\frac{\partial}{\partial\varphi}\left(\frac{\alpha}{r}\right)+\frac{\partial}{\partial\zeta}\left(\rho\dot{\zeta}\alpha\right)=\alpha F_\rho \tag{A.17}$$

let $\rho^*=\rho\frac{r^2}{a^2}\frac{\partial r}{\partial\zeta}$ and put it into the above equation, we get the density equation in generalized coordinates as

$$\frac{\partial\rho^*}{\partial t}+\frac{\partial}{\partial\lambda}\left(\rho^*\dot{\lambda}\right)+\frac{\partial}{\partial\mu}\left(\rho^*\dot{\mu}\right)+\frac{\partial}{\partial\zeta}\left(\rho^*\dot{\zeta}\right)=F_\rho^* \tag{A.18}$$

Thus, the cosine is removed and the density is more like a height-weighted density.

Appendix B
The base state for linearization

To linearize the discretized equations, we have to define a base state, as usually used in the NCEP GFS, which is a rest atmosphere with a given balanced hydrostatic state. Let's start by defining following constants

$$\hat{p}_{01} = 101.326 \tag{B.1}$$

$$\hat{r}_{01} = a = 6371220 \tag{B.2}$$

and the given thermodynamic constants for

$$T_{0k} \tag{B.3}$$

which can be a constant (300K) or a function of a given lapse rate. And the related constants can be listed as follows

$$h_{0k} = C_{P_{0k}} T_{0k} \tag{B.4}$$

$$\kappa_{0k} = R_{0k} / C_{P_{0k}} \tag{B.5}$$

Then the base coordinate pressures at all levels can be defined by the base pressure at the first level as

$$\hat{\tilde{\tilde{p}}}_{0k} = \hat{A}_k + \hat{B}_k \hat{p}_{01} \tag{B.6}$$

And from discretized form of Eq. (4.5)

$$\frac{\Delta \tilde{\tilde{p}}_{0k}}{\Delta r_{0k}} = \frac{\hat{\tilde{\tilde{p}}}_{0k} - \hat{\tilde{\tilde{p}}}_{0k+1}}{\hat{r}_{0k} - \hat{r}_{0k+1}} = -\rho_{0k} \overline{g} \frac{r_{0k}^2}{a^2} \tag{B.7}$$

and base hydrostatic state

$$\frac{\Delta p_{0k}}{\Delta r_{0k}} = \frac{\hat{p}_{0k} - \hat{p}_{0k+1}}{\hat{r}_{0k} - \hat{r}_{0k+1}} = -\rho_{0k} \overline{g} \tag{B.8}$$

we have a relation between base pressure and base coordinate pressure as

$$\Delta p_{0k} = \Delta \tilde{\tilde{p}}_{0k} \frac{a^2}{r_{0k}^2} \tag{B.9}$$

To obtain base pressure, let's have the first guess $\Delta p_{0k} = \Delta \tilde{\tilde{p}}_{0k}$ and use Eq. (B.8) to obtain

$$\hat{r}_{0k+1} = \hat{r}_{0k} + 2 \frac{\hat{p}_{0k} - \hat{p}_{0k+1}}{\hat{p}_{0k} + \hat{p}_{0k+1}} \frac{\kappa_{0k} h_{0k}}{\overline{g}} \tag{B.10}$$

And value at model layers as

$$r_{0k}^3 = \frac{1}{2}\left(\hat{r}_{0k}^3 + \hat{r}_{0k+1}^3\right) \tag{B.11}$$

Then we use r in Eq. (B.11) to obtain new Δp_{0k} by Eq. (B.9), then repeat through (B.10) and (B.11) to get convergence of Δp_{0k}. Thus, we have all base fields for linearization.

Appendix C
Matrices for semi-implicit scheme

We can summarize all linear terms in Section 6 with the definition in Section 7 shown here with their matrices and vectors for the semi-implicit scheme as

$$\left(\frac{D\tilde{\tilde{p}}_s}{Dt}\right)_L = -\Pi_{1k} D_k^* \tag{C.1}$$

$$\left(\frac{d\tilde{w}_k}{dt}\right)_L = -b_{0k}\tilde{\tilde{p}}_s + \Gamma_{ki} p_i \tag{C.2}$$

$$\left(\frac{dp_k}{dt}\right)_L = -d_{0k} D_k^* + \mathrm{M}_{ki}\tilde{w}_i \tag{C.3}$$

$$\left(\frac{dh_k}{dt}\right)_L = -f_{0k} D_k^* + \mathrm{Z}_{ki}\tilde{w}_i \tag{C.4}$$

$$\left(\frac{dD_k^*}{dt}\right)_L = -\mathrm{A}_{ki}\nabla^2 p_i - \mathrm{B}_{ki}\nabla^2 h_i - e_k \nabla^2 p_s \tag{C.5}$$

Here we use only 6 layers for an example. In order to have the same vertical index direction as in the model, as in Fig. 1, we make the conventional matrix index in following way, for any given matrix B, as

$$\mathrm{B} = \begin{bmatrix} B_{66} & B_{65} & B_{64} & B_{63} & B_{62} & B_{61} \\ B_{56} & B_{55} & B_{54} & B_{53} & B_{52} & B_{51} \\ B_{46} & B_{45} & B_{44} & B_{43} & B_{42} & B_{41} \\ B_{36} & B_{35} & B_{34} & B_{33} & B_{32} & B_{31} \\ B_{26} & B_{25} & B_{24} & B_{23} & B_{22} & B_{21} \\ B_{16} & B_{15} & B_{14} & B_{13} & B_{12} & B_{11} \end{bmatrix} \tag{C.6}$$

Then the matrix P, for converting differences in pressure at neighboring model levels to the pressure at a model layer is as follows

$$\mathrm{P} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0.5 \end{bmatrix} \tag{C.7}$$

the same for matrix W, for converting the difference in height-weighted coordinate vertical velocity at neighboring model levels to a model layer as

$$W=\begin{bmatrix} 0.5 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.5 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \tag{C.8}$$

We need to do a matrix invert for P and W in following matrices. Let's start by listing matrices, vectors, and constants for all linear equations from (C.1) to (C.5).

The vector for Eq. (C.1) is

$$\Pi=\begin{pmatrix} \dfrac{\Delta\tilde{\bar{p}}_{06}}{\varepsilon_{06}} & \dfrac{\Delta\tilde{\bar{p}}_{05}}{\varepsilon_{05}} & \dfrac{\Delta\tilde{\bar{p}}_{04}}{\varepsilon_{04}} & \dfrac{\Delta\tilde{\bar{p}}_{03}}{\varepsilon_{03}} & \dfrac{\Delta\tilde{\bar{p}}_{02}}{\varepsilon_{02}} & \dfrac{\Delta\tilde{\bar{p}}_{01}}{\varepsilon_{01}} \end{pmatrix} \tag{C.9}$$

The matrix for Eq. (C.2) with inverted matrix P is

$$\Gamma=\begin{matrix} \bar{g}\varepsilon_{06}^{4}/\Delta\tilde{\bar{p}}_{06} \\ \bar{g}\varepsilon_{05}^{4}/\Delta\tilde{\bar{p}}_{05} \\ \bar{g}\varepsilon_{04}^{4}/\Delta\tilde{\bar{p}}_{04} \\ \bar{g}\varepsilon_{03}^{4}/\Delta\tilde{\bar{p}}_{03} \\ \bar{g}\varepsilon_{02}^{4}/\Delta\tilde{\bar{p}}_{02} \\ \bar{g}\varepsilon_{01}^{4}/\Delta\tilde{\bar{p}}_{01} \end{matrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0.5 \end{bmatrix}^{-1} \tag{C.10}$$

The constant for Eq. (C.2) is

$$b_{0k}=\frac{\bar{g}}{\Delta\tilde{\bar{p}}_{0k}}\Delta B_k \tag{C.11}$$

The matrix M for Eq. (C.3) follows with inverted matrix W

$$\mathrm{M}=\begin{matrix} \gamma_{06}\bar{g}p_{06}^{2}/\left(\kappa_{06}h_{06}\Delta\tilde{\bar{p}}_{06}\right) \\ \gamma_{05}\bar{g}p_{05}^{2}/\left(\kappa_{05}h_{05}\Delta\tilde{\bar{p}}_{05}\right) \\ \gamma_{04}\bar{g}p_{04}^{2}/\left(\kappa_{04}h_{04}\Delta\tilde{\bar{p}}_{04}\right) \\ \gamma_{03}\bar{g}p_{03}^{2}/\left(\kappa_{03}h_{03}\Delta\tilde{\bar{p}}_{03}\right) \\ \gamma_{02}\bar{g}p_{02}^{2}/\left(\kappa_{02}h_{02}\Delta\tilde{\bar{p}}_{02}\right) \\ \gamma_{01}\bar{g}p_{01}^{2}/\left(\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}\right) \end{matrix} \begin{bmatrix} 0.5 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.5 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}^{-1} \tag{C.12}$$

the constant for Eq. (C.3) is

$$d_{0k}=\frac{\gamma_{0k}p_{0k}}{\varepsilon_{0k}} \tag{C.13}$$

The matrix Z for Eq. (C.4) with inverted matrix W is

$$
Z = \begin{array}{c} \gamma_{06}\bar{g}p_{06} / \Delta\tilde{\bar{p}}_{06} \\ \gamma_{05}\bar{g}p_{05} / \Delta\tilde{\bar{p}}_{05} \\ \gamma_{04}\bar{g}p_{04} / \Delta\tilde{\bar{p}}_{04} \\ \gamma_{03}\bar{g}p_{03} / \Delta\tilde{\bar{p}}_{03} \\ \gamma_{02}\bar{g}p_{02} / \Delta\tilde{\bar{p}}_{02} \\ \gamma_{01}\bar{g}p_{01} / \Delta\tilde{\bar{p}}_{01} \end{array}
\begin{bmatrix} 0.5 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0.5 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}^{-1} \tag{C.14}
$$

The constant for Eq. (C.4) is

$$
f_{0k} = \frac{\gamma_{0k}\kappa_{0k}h_{0k}}{\varepsilon_{0k}} \tag{C.15}
$$

Then the matrix A for Eq. (C.5) is

$$
\mathrm{A} = - \begin{array}{c} \dfrac{\Delta p_{06}}{\Delta\tilde{\bar{p}}_{06}\varepsilon_{06}} \\[2ex] \dfrac{\Delta p_{05}}{\Delta\tilde{\bar{p}}_{05}\varepsilon_{05}} \\[2ex] \dfrac{\Delta p_{04}}{\Delta\tilde{\bar{p}}_{04}\varepsilon_{04}} \\[2ex] \dfrac{\Delta p_{03}}{\Delta\tilde{\bar{p}}_{03}\varepsilon_{03}} \\[2ex] \dfrac{\Delta p_{02}}{\Delta\tilde{\bar{p}}_{02}\varepsilon_{02}} \\[2ex] \dfrac{\Delta p_{01}}{\Delta\tilde{\bar{p}}_{01}\varepsilon_{01}} \end{array}
\begin{bmatrix}
\dfrac{\kappa_{06}h_{06}\Delta\tilde{\bar{p}}_{06}}{2p_{06}^2} & \dfrac{\kappa_{05}h_{05}\Delta\tilde{\bar{p}}_{05}}{p_{05}^2} & \dfrac{\kappa_{04}h_{04}\Delta\tilde{\bar{p}}_{04}}{p_{04}^2} & \dfrac{\kappa_{03}h_{03}\Delta\tilde{\bar{p}}_{03}}{p_{03}^2} & \dfrac{\kappa_{02}h_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \dfrac{\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & \dfrac{\kappa_{05}h_{05}\Delta\tilde{\bar{p}}_{05}}{2p_{05}^2} & \dfrac{\kappa_{04}h_{04}\Delta\tilde{\bar{p}}_{04}}{p_{04}^2} & \dfrac{\kappa_{03}h_{03}\Delta\tilde{\bar{p}}_{03}}{p_{03}^2} & \dfrac{\kappa_{02}h_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \dfrac{\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & \dfrac{\kappa_{04}h_{04}\Delta\tilde{\bar{p}}_{04}}{2p_{04}^2} & \dfrac{\kappa_{03}h_{03}\Delta\tilde{\bar{p}}_{03}}{p_{03}^2} & \dfrac{\kappa_{02}h_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \dfrac{\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & 0 & \dfrac{\kappa_{03}h_{03}\Delta\tilde{\bar{p}}_{03}}{2p_{03}^2} & \dfrac{\kappa_{02}h_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \dfrac{\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & 0 & 0 & \dfrac{\kappa_{02}h_{02}\Delta\tilde{\bar{p}}_{02}}{2p_{02}^2} & \dfrac{\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & 0 & 0 & 0 & \dfrac{\kappa_{01}h_{01}\Delta\tilde{\bar{p}}_{01}}{2p_{01}^2}
\end{bmatrix} \tag{C.16}
$$

$$
+ \begin{bmatrix}
\dfrac{k_{06}h_{06}}{p_{06}\varepsilon_{06}} & 0 & 0 & 0 & 0 & 0 \\
0 & \dfrac{k_{05}h_{05}}{p_{05}\varepsilon_{05}} & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{k_{04}h_{04}}{p_{04}\varepsilon_{04}} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{k_{03}h_{03}}{p_{03}\varepsilon_{03}} & 0 & 0 \\
0 & 0 & 0 & 0 & \dfrac{k_{02}h_{02}}{p_{02}\varepsilon_{02}} & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{k_{01}h_{01}}{p_{01}\varepsilon_{01}}
\end{bmatrix}
$$

and the matrix B for Eq. (C.5) is the following

$$
B = \begin{array}{c} \frac{\Delta p_{06}}{\Delta\tilde{\bar{p}}_{06}\varepsilon_{06}} \\ \frac{\Delta p_{05}}{\Delta\tilde{\bar{p}}_{05}\varepsilon_{05}} \\ \frac{\Delta p_{04}}{\Delta\tilde{\bar{p}}_{04}\varepsilon_{04}} \\ \frac{\Delta p_{03}}{\Delta\tilde{\bar{p}}_{03}\varepsilon_{03}} \\ \frac{\Delta p_{02}}{\Delta\tilde{\bar{p}}_{02}\varepsilon_{02}} \\ \frac{\Delta p_{01}}{\Delta\tilde{\bar{p}}_{01}\varepsilon_{01}} \end{array}
\begin{bmatrix}
\frac{\kappa_{06}\Delta\tilde{\bar{p}}_{06}}{2p_{06}^2} & \frac{\kappa_{05}\Delta\tilde{\bar{p}}_{05}}{p_{05}^2} & \frac{\kappa_{04}\Delta\tilde{\bar{p}}_{04}}{p_{04}^2} & \frac{\kappa_{03}\Delta\tilde{\bar{p}}_{03}}{p_{03}^2} & \frac{\kappa_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \frac{\kappa_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & \frac{\kappa_{05}\Delta\tilde{\bar{p}}_{05}}{2p_{05}^2} & \frac{\kappa_{04}\Delta\tilde{\bar{p}}_{04}}{p_{04}^2} & \frac{\kappa_{03}\Delta\tilde{\bar{p}}_{03}}{p_{03}^2} & \frac{\kappa_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \frac{\kappa_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & \frac{\kappa_{04}\Delta\tilde{\bar{p}}_{04}}{2p_{04}^2} & \frac{\kappa_{03}\Delta\tilde{\bar{p}}_{03}}{p_{03}^2} & \frac{\kappa_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \frac{\kappa_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & 0 & \frac{\kappa_{03}\Delta\tilde{\bar{p}}_{03}}{2p_{03}^2} & \frac{\kappa_{02}\Delta\tilde{\bar{p}}_{02}}{p_{02}^2} & \frac{\kappa_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & 0 & 0 & \frac{\kappa_{02}\Delta\tilde{\bar{p}}_{02}}{2p_{02}^2} & \frac{\kappa_{01}\Delta\tilde{\bar{p}}_{01}}{p_{01}^2} \\
0 & 0 & 0 & 0 & 0 & \frac{\kappa_{01}\Delta\tilde{\bar{p}}_{01}}{2p_{01}^2}
\end{bmatrix} \tag{C.17}
$$

finally, the constant for Eq. (C.5) is

$$
e_{0k} = \frac{\Delta p_{0k}}{\Delta\tilde{\bar{p}}_{0k}} \frac{1}{\varepsilon_{0k}} \left\langle \frac{\kappa_k h_k \Delta B_k}{2p_{0k}} + \sum_{i=1}^{k-1} \frac{\kappa_{0i} h_{0i} \Delta B_{0i}}{p_{0i}} \right\rangle \tag{C.18}
$$

All these are prepared initially and used to construct further matrices for the semi-implicit computation.

Appendix D
Initial conditions for a cold start from the hydrostatic system

Initial conditions from hydrostatic system will require the generation of vertical velocity and a vertical interpolation from the pressure defined hydrostatic system to a coordinate pressure defined deep atmospheric nonhydrostatic system. The pressure in deep-atmospheric nonhydrostatic system is also embedded in the relationship between coordinate pressure and height.

From our current hydrostatic data system, we have pressure defined as

$$\hat{p}_k = \hat{A}_k + \hat{B}_k p_s \tag{D.1}$$

and

$$p_k = \frac{1}{2}\left(\hat{p}_k + \hat{p}_{k+1}\right) \tag{D.2}$$

and for hydrostatic relationship in the hydrostatic system, we have

$$\hat{z}_{k+1} = \hat{z}_k + \left(\hat{p}_k - \hat{p}_{k+1}\right)\frac{\kappa_k h_k}{p_k \overline{g}} \tag{D.3}$$

so the coordinate pressure in the hydrostatic system can be computed based on Eq. (4.15) as

$$\left(\hat{\overline{\overline{p}}}_k\right)_{hyd} = \left(\hat{\overline{\overline{p}}}_{k+1}\right)_{hyd} + \frac{\left(\left(\hat{z}+a\right)^3_{k+1} - \left(\hat{z}+a\right)^3_k\right)}{3a^2}\left(\frac{p\overline{g}}{\kappa h}\right)_k \tag{D.4}$$

with zero at top of coordinate pressure, and the coordinate pressure at ground surface becomes

$$\hat{\overline{\overline{p}}}_s = \left(\hat{\overline{\overline{p}}}_s\right)_{hyd} = \sum_{k=1}^{K}\frac{\hat{r}^3_{k+1} - \hat{r}^3_k}{3a^2}\left(\frac{p\overline{g}}{\kappa h}\right)_k \tag{D.5}$$

Then the coordinate pressure in deep-atmospheric nonhydrostatic system is defined by Eq. (6.13) to obtain new coordinates defined by coordinate pressure. So the prognostic variables in hydrostatic system can be interpolated to a deep atmospheric nonhydrostatic system by the coordinate depth between Eq. (D.4) and Eq. (6.13). The prognostic variables in this interpolation can be three dimensional momentum, enthalpy, and tracers. Then the iteration process in Appendix B is used to obtain regular pressure, which satisfies hydrostatic relationship and balanced with coordinate pressure.

We can obtain vertical motion from the hydrostatic system then interpolate as we mentioned in the last paragraph, or we can obtain it after interpolation in the deep atmospheric nonhydrostatic system. To get vertical motion in the hydrostatic system, we can do a total derivative for Eq. (D.3) and get

$$\hat{w}_{k+1} = \hat{w}_k + \frac{\kappa_k h_k}{p_k g}\frac{d\Delta p_k}{dt} - \Delta p_k \frac{\kappa_k h_k}{\gamma p_k^2 g}\frac{dp_k}{dt} \tag{D.6}$$

and from Eq. (6.7), the above equation can be written as

$$\hat{w}_{k+1} = \hat{w}_k + \frac{\kappa_k h_k}{p_k g}\left(\mathrm{P}_{ki}^{-1} \frac{dp_i}{dt} - \frac{\Delta p_k}{\gamma p_k} \frac{dp_k}{dt} \right) \tag{D.7}$$

From the hydrostatic system, see Juang (2011) Eq. (B.3), we know

$$\frac{dp_k}{dt} = \frac{m^2}{2}\left\{ V_k^* \bullet \nabla(\hat{p}_k + \hat{p}_{k+1}) - \sum_{i=k}^{K}\left(\Delta p_i D_i^* + V_i^* \bullet \nabla \Delta p_i\right) - \sum_{i=k+1}^{K}\left(\Delta p_i D_i^* + V_i^* \bullet \nabla \Delta p_i\right) \right\} \tag{D.8}$$

and boundary conditions for Eq. (D.7) is

$$\hat{w}_1 = m^2 \left(u_1^* \frac{\partial z_s}{a \partial \lambda} + v_1^* \frac{\partial z_s}{a \partial \varphi} \right) \tag{D.9}$$

and all vertical motion at model layers is given as

$$w_k = \frac{1}{2}\left(\hat{w}_k + \hat{w}_{k+1} \right) \tag{D.10}$$

Then the vertical motion is interpolated from the hydrostatic system to deep atmosphere nonhydrostatic system as mentioned.

However, to get vertical motion in a deep atmospheric nonhydrostatic system with other interpolated prognostic variables, we can start with Eq. (5.11) as a total derivative as

$$\hat{\tilde{w}}_{k+1} = \hat{\tilde{w}}_k + \frac{\kappa_k h_k}{p_k g} \Delta B_k \frac{d\hat{\tilde{\bar{p}}}_s}{dt} - \Delta \tilde{\bar{p}}_k \frac{\kappa_k h_k}{\gamma p_k^2 g} \frac{dp_k}{dt} \tag{D.11}$$

where the second term on the RHS can be given by Eq. (4.11) and the third term on the RHS is given by Eq. (5.26). Then the coordinate vertical motion at a model layer is given by Eq. (5.37) and the bottom condition by Eq. (5.38).

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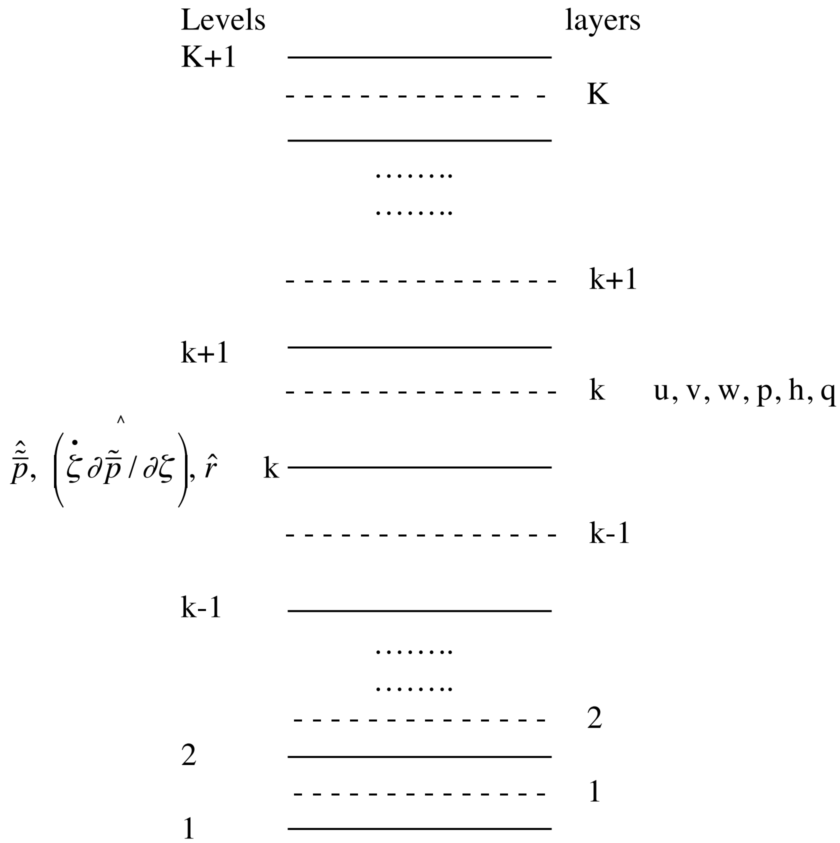

Fig. 1 The vertical grid structure with layers and levels is used in discretization. Integers are used to index the layers and levels. Variables with a hat are on levels, and without a hat are on layers.